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ASTRONOMY AND ASTROPHYSICS

Vertical distribution of temperature and density in a planetary ring

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Abstract. We model temperature and density profiles for a dilute planetary ring, based on the hydrodynamic balance equations for momentum and energy of granular flows. Within our approximation the ring consists of inelastic smooth spheres of unique size and mass, while the fluxes of mass, momentum and energy are linear functions of the gradients of density, velocity and temperature. The resulting system of coupled differential equations leads to temperature and density profiles, which we compare to the results of a triaxial kinetic approach to the dynamics of a planetary ring. We find that both approaches agree fairly well in the elastic limit. Further, we carry out event driven N-particle simulations of a ring, subject to the conditions of the theoretical model. The simulated profiles are then compared to the theoretical ones: for the density a good agreement is found for both theoretical approaches, but the simulated temperature profiles increase monotonically with vertical distance whereas the theoretical profiles always have a turnover at some distance from the mid plane. This disagreement is likely to be connected to the vertical dependence of the velocity ellipsoid, which is not taken into account in the theoretical treatments.

Key words: hydrodynamics – methods: N-body simulations – planets and satellites: general

1. Introduction

In the past, theoretical investigations as well as simulations of planetary rings have mainly concentrated on the disk's horizontal features, e.g. gaps, wakes or resonances, (see for instance Showalter et al. (1986) and references therein). In the majority of the applications the ring was treated as a two-dimensional object, an approximation that is justified for many purposes, considering for example the thickness of Saturn's ring system ($\sim 10\mathrm{m}-100\mathrm{m}$) of only a few typical particle diameters compared to its horizontal extension ($\sim 10^8\mathrm{m}$). On the other hand there are dynamical properties of the particle ensemble, such as the collision frequency and the optical depth, that clearly possess their particular functional dependence on density and temperature due to the rings' actual three-dimensional nature (Stewart et al., 1984). Therefore, a knowledge of the vertical ring structure is of principal interest.

Theories describing the ring dynamics usually incorporate methods from kinetic theory, in order to derive balance laws for mass, mean velocity and energy of the particle stream. In the literature the non-isotropic nature of a ring system is taken into account by using a triaxial Gaussian velocity distribution (Goldreich & Tremaine, 1978; Araki & Tremaine, 1986; Araki, 1988; Araki, 1991). Within this framework the description of the non-equilibrium state of the system is of zeroth order, in the sense of a systematic expansion of the distribution function in orders of the gradients of density, mean velocity and temperature. In the same theoretical context Simon & Jenkins (1994) studied the vertical components of the balance laws for mass, momentum, and energy, and obtained numerical solutions for the temperature and density, depending on the height above the midplane of the planetary ring.

In this study, we follow the usual hydrodynamic approach to granular kinetics and compare the results to those of Simon and Jenkins, i.e. we employ an isotropic Gaussian phase space distribution plus first order corrections to give a consistent description of the non-equilibrium fluxes of mass, momentum, and energy. The particles are identical smooth spheres (i.e. rotational degrees of freedom are neglected) that collide inelastically, where the energy dissipation is described in terms of a constant normal coefficient of restitution. We make use of constitutive relations derived by Jenkins & Richman (1985), where kinetic theory has been systematically extended to include the dissipative nature of the interparticle collisions in granular matter (see also Lun et al. 1984). In the present work, we employ their results in order to describe the ring as a (three-dimensional) granular flow under the influence of the gravity of the central planet.

Under the same conditions we then make simulations of a planetary ring, varying the optical depth and the coefficient of restitution. We use an event driven code to simulate N particles in a box in the gravitational field of a central mass with periodic boundary conditions in radial and azimuthal direction. We compare the vertical stratification of density and temperature in the simulation box to the theoretical profiles of the hydrody-

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namic and the triaxial kinetic approach to the dynamics of the ensemble of inelastic ring particles.

The paper is organized as follows: In Sect. 2 the balance equations for mass, momentum, and energy are given, derived by using an isotropic Gaussian distribution function, together with the constitutive relations. We then solve the vertical components of the balance laws numerically. Furthermore, we resolve the corresponding balance equations of the triaxial approach of Simon & Jenkins (1994) and discuss the differences to the solutions of the hydrodynamic approach. In Sect. 3 we present the simulations and the comparison to theoretical expressions. Finally, in Sect. 4 we summarize our results and discuss the solutions.

2. Balance laws

2.1. Hydrodynamic approach

The dynamics of granular matter are theoretically described in terms of kinetic theory, based upon a kinetic equation of Boltzmann or Enskog type. By computing the first three moments of this kinetic equation, the balance laws of mass, momentum, and energy for a dilute system of inelastic smooth spheres read

$$\begin{aligned}(\partial_t + u_\alpha\nabla_\alpha)\varrho &= -\varrho\nabla_\alpha u_\alpha \\ \varrho(\partial_t + u_\alpha\nabla_\alpha)u_\beta &= \varrho F_\beta - \nabla_\alpha P_{\alpha\beta} \\ \frac{3}{2}\varrho(\partial_t + u_\alpha\nabla_\alpha)T &= -P_{\alpha\beta}\nabla_\alpha u_\beta - \nabla_\alpha q_\alpha - \Gamma\,.\end{aligned} \tag{1}$$

Here ϱ and $\boldsymbol{u}$ are the density and the mean velocity of the ring material and $\boldsymbol{F}$ is the external force exerted on the particles. $T = (1/3)\langle(\boldsymbol{v}-\boldsymbol{u})^2\rangle$ is the granular temperature ($m := 1, k_B := 1$) and $\boldsymbol{q} = (1/2)\varrho\langle(\boldsymbol{v}-\boldsymbol{u})(\boldsymbol{v}-\boldsymbol{u})^2\rangle$ is the heat flux, defined in terms of the velocity fluctuations, $\hat{\boldsymbol{P}}$ is the pressure tensor, and Γ is the steady dissipation of energy due to inelastic interparticle collisions. We relate the thermodynamic fluxes of mass ($\varrho\boldsymbol{u}$), momentum ($\hat{\boldsymbol{P}}$), and energy ($\boldsymbol{q}$) to their respective thermodynamic forces ($\boldsymbol{\nabla}\varrho$, $\boldsymbol{\nabla}\boldsymbol{u}$, $\boldsymbol{\nabla}T$) via a linear ansatz, i.e.

$$P_{\alpha\beta} = p\delta_{\alpha\beta} - 2\eta D_{\alpha\beta} - \zeta\nabla_\mu u_\mu\delta_{\alpha\beta} \tag{2}$$

and

$$q_\alpha = -\kappa\nabla_\alpha T\,. \tag{3}$$

The shear and bulk viscosity and the heat conductivity are denoted by η, ζ and κ respectively, and $\hat{\boldsymbol{D}}$ is the shear tensor.

$$D_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\nabla_\lambda u_\lambda)\,. \tag{4}$$

For the transport coefficients κ and η, as well as for the cooling Γ we employ the expressions derived by Jenkins & Richman (1985) for granular gases in the dilute and nearly elastic limit

$$\begin{aligned}\eta &= \frac{5}{24}\frac{\sqrt{\pi}\varrho_p\,d}{(1+\epsilon)(3-\epsilon)}\sqrt{T} \\ \kappa &= \frac{25}{4}\frac{\sqrt{\pi}\varrho_p\,d}{(1+\epsilon)(49-33\epsilon)}\sqrt{T} \\ \Gamma &= 12\frac{(1-\epsilon^2)\nu^2\varrho_p}{\sqrt{\pi}d}\sqrt{T^3}\,.\end{aligned} \tag{5}$$

Here ϵ is the constant coefficient of restitution, ϱ_p and d are the particle's bulk density and diameter, respectively, and $\nu = \varrho/\varrho_p$ is the filling factor. We neglect corrections to the transport coefficients and the equation of state (Chapman & Cowling, 1970; Canahan & Starling, 1969) due to the finite particle volume, keeping in mind that they become important in the denser regions of a planetary ring (Araki & Tremaine, 1986).

In order to apply Eq. (1) to the particle flow in a planetary ring, we choose cylindrical coordinates (r, φ, z), where $z = 0$ is the midplane of the ring. The external force $\boldsymbol{F}$ is the gravitation of the planet

$$\boldsymbol{F} = \boldsymbol{\nabla}\frac{GM}{\sqrt{r^2+z^2}} = -\Omega^2(\boldsymbol{e_r}r + \boldsymbol{e_z}z) + O\{(z/r)^2\} \tag{6}$$

with the Kepler frequency $\Omega = \sqrt{GM/r^3}$. In the following we neglect orders of $(z/r)^2$, since r is about 10^8m, while we are interested in a vertical extension of less than a hundred meters. Furthermore, for a stationary ring we have $\boldsymbol{u} = r\Omega\boldsymbol{e_\varphi}$, $\partial_t\{\varrho, \boldsymbol{u}, T\} = 0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. We want to obtain the vertical profiles of density and temperature, thus we evaluate the z-components of the balance laws (1) subject to the latter conditions. Together with the constitutive relations (2) and (3) and the transport coefficients (5) we find

$$\begin{aligned}(\nu T)' + z\Omega^2\nu &= 0 \\ a(\epsilon)\left[T'\sqrt{T}\right]' + b(\epsilon)\Omega^2\sqrt{T} - \frac{c(\epsilon)}{d^2}\nu^2\sqrt{T^3} &= 0\end{aligned} \tag{7}$$

for the balance of momentum and energy, while the continuity equation is trivially fulfilled. A prime denotes differentiation with respect to z. In the second equation the first term stands for the contribution of heat conductivity to the vertical energy balance, the second one describes viscous heating, and the third the collisional cooling. The functions a, b and c stand for their respective dependences on the coefficient of restitution, given by

$$\begin{aligned}a &= \frac{1}{(1+\epsilon)(49-33\epsilon)} \\ b &= \frac{3}{40(1+\epsilon)(3-\epsilon)} \\ c &= \frac{48}{25\pi}(1-\epsilon^2)\,.\end{aligned} \tag{8}$$

The calculation of the transport coefficients (5) is restricted to the nearly elastic case and further it is based on the assumption of small gradients of temperature, density, and velocity. So, for instance, the functional dependence of the viscosity in (5) on the temperature and in particular its independence of the density are valid only if the systems density and temperature can be considered as being constant over distances larger than the mean free path of the particles. Furthermore, since any ring particle moves on a Keplerian orbit in between two collisions, its free path is essentially restricted by its epicyclic motion (Goldreich & Tremaine, 1978; Stewart et al., 1984). Strictly speaking, the collision time of the hydrodynamic system

$$t_c = \sqrt{\pi}d/(24\sqrt{T}\nu) \tag{9}$$

(see Chapman & Cowling 1970) should be significantly smaller than the characteristic time scale of Keplerian motion $t_k \sim \pi/\Omega$, i.e.

$$\frac{\Omega d}{\nu\sqrt{T}} < 24\sqrt{\pi}\,. \tag{10}$$

Assuming for the moment a Gaussian density profile $\nu = \nu_0 \exp\{-(z/H)^2\}$ with a typical scale height H and a constant (i.e. z-independent) temperature T_0 (see Stewart et al. 1984), we find $z < 2H$, for $T_0 \sim 10^{-4}\mathrm{m}^2/\mathrm{s}^2, \nu_0 \sim 10^{-2}, \Omega \sim 10^{-4}\mathrm{s}^{-1}$, and $d \sim 1\mathrm{m}$. This means that the validity of (5) is restricted to the region near the midplane of the ring. We have to be aware of this fact when applying (7) to the ring system, where the density evidently decreases quickly out of the equatorial plane. Further, since the impact frequency scales proportional to the optical depth τ, we expect inequality (10) to be valid up to higher latitudes for the case of a larger optical depth τ.

2.2. Comparison between hydrodynamic and triaxial kinetic approach

In the framework of the triaxial approach of Goldreich & Tremaine (1978) to ring dynamics, Simon & Jenkins (1994) investigated the vertical structure of a planetary ring. They calculated the restitution dependence of the second moment of the velocity fluctuations approximatively and formulated the balance laws in terms of an anisotropic Gaussian velocity distribution. Under the assumption of a *z-independent* velocity ellipsoid the vertical components of these balance laws lead to two coupled differential equations for density and temperature ($T \equiv 1/3\,\mathrm{Tr}\{T_{\alpha\beta}\}$) that are the triaxial analogue of Eq. (7):

$$\begin{aligned} (\nu T)' + \lambda(\epsilon) z \Omega^2 \nu &= 0 \\ \alpha(\epsilon)\left[T'\sqrt{T}\right]' + \frac{\beta(\epsilon)}{d}\Omega T\nu - \frac{\gamma(\epsilon)}{d^2}\nu^2\sqrt{T^3} &= 0 \end{aligned} \tag{11}$$

with

$$\begin{aligned} \alpha &= \frac{[1-\frac{2}{7}(1-\epsilon)][\frac{1}{5}-\frac{1}{7}(1-\epsilon)]}{(1+\epsilon)\left\{49-33\epsilon+\frac{1-\epsilon}{2744}(5\epsilon-54)(237\epsilon-461)\right\}} \\ \beta &= \frac{1}{9-5\epsilon}\sqrt{\frac{1-\epsilon}{125\pi}\,(225\epsilon^2+4415\epsilon-2876)} \\ \gamma &= \frac{48}{25\pi}(1+\frac{(\epsilon-1)(195\epsilon-979)}{6272})(1-\epsilon^2) \end{aligned} \tag{12}$$

and

$$\lambda = \frac{7}{2+5\epsilon}\,.$$

We observe a different temperature and density dependence of the heating term ($\sim T\nu$) in comparison to Eq. (7) ($\sim \sqrt{T}$). In the hydrodynamic description the heating is a consequence of the granular viscosity, which is a material property of the ring particles. As already argued, the mean free path of a particle is restricted by the epicyclic motion. If $\bar{v}$ denotes the most probable velocity of the particle and l is its mean free path, we have $\eta \sim \nu\bar{v}l$. As long as the collision time t_c (Eq. (9)) is substantially smaller than the characteristic orbital time $t_k = \pi/\Omega$ we have $l = \bar{v}t_c$ and thus $\eta = \nu\bar{v}^2 t_c$. For inelastic particles we then obtain the formula (5), since $\bar{v} \sim \sqrt{T}$, and the density dependence cancels. For low optical depths and for higher latitudes however, we should set $l = \bar{v}t_k$, since the mean free path is restricted by the epicyclic motion. Then the density dependence of the heating term is retained and we find $\eta \sim T\nu$. In detail this modified viscosity η^* reads

$$\eta^* = 5\frac{\pi\varrho_p}{\Omega(1+\epsilon)(3-\epsilon)}\nu T\,. \tag{13}$$

We will apply this modification in Sect. 3 and discuss its improvement to the hydrodynamic balance laws, when we compare the theoretical and simulated stratification.

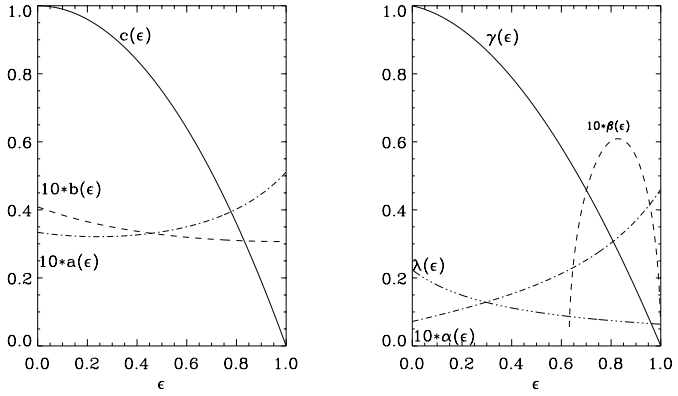

Fig. 1. Restitution dependence of the coefficients in Eq. (7) and (11). a, b, c, and α, β, γ stand for the dependence on ϵ of the heat conduction, heating, and collisional cooling, respectively. The function λ is a measure for the restitution dependence of the temperature anisotropy. (All shown quantities are normalized by the maximum values of c or γ, respectively.)

Comparing the coefficients a, b, c with α, β, γ that describe the ϵ dependence of the heat flux, heating, and cooling contribution to the energy balance, respectively (see Fig. 1), we find rough qualitative and quantitative agreement for the cooling in the triaxial and the hydrodynamic system. In both cases the cooling vanishes, when the coefficient of restitution approaches unity. However, we observe different properties for the heating: in the hydrodynamic approach we see a finite heating contribution for all values of the restitution due to the granular viscosity, monotonically increasing as ϵ decreases. On the other hand, the heating contribution to the triaxial heat balance equation is a consequence of the temperature anisotropy, which in turn stems from the inelasticity of the interparticle collisions. Thus, the heating term becomes zero for elastic particles ($\epsilon = 1$), where the temperature is isotropic. In this case, since the distribution function is purely Gaussian, the heat balance equation cannot contain a heating term. The critical value $\epsilon_c \approx 0.635$, where β approaches zero, corresponds in the z-integrated triaxial kinetic approach to ring dynamics to a vanishing optical depth τ, thus, in this limit, heating is getting more and more inefficient.

We observe an additional restitution dependence $\lambda(\epsilon)$ of the momentum balance equation (first relation in Eq. (11)), which is related to the temperature anisotropy $T_{zz} < T$ (Simon & Jenkins, 1994).

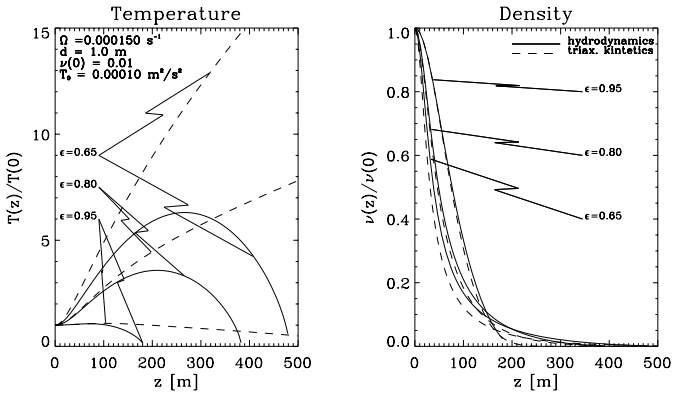

Fig. 2. Temperature and density profiles from the numerical solution of the linear hydrodynamic (7) and the triaxial kinetic balance equations (11) for different values of the coefficient of restitution.

We note that for a zero order distribution function, like the anisotropic Gaussian, all odd moments of the velocity fluctuation $(v_\alpha - u_\alpha)$ vanish. Consequently, the heat flux ($\sim (v_\alpha - u_\alpha)(\boldsymbol{v}-\boldsymbol{u})\cdot(\boldsymbol{v}-\boldsymbol{u})$) is *a priori* zero (Zhang, 1993). Simon & Jenkins (1994) provided a heat conduction contribution to their energy balance equation, by incorporating a form of the heat flux calculated by Zhang (1993), which was derived using higher order corrections to the anisotropic Gaussian.

2.3. Numerical Solution

Next, we solve Eqs. (7) and (11) numerically, employing a variable step size fourth order Runge Kutta method. Since the differential equations are first order in ν and second order in T, the appropriate initial conditions are

$$T(0) = T_0 \quad \nu(0) = \nu_0 \quad T'(0) = 0$$

where the last relation is due to the requirement $T(z) = T(-z)$.

If we choose as parameter values $\Omega = 1.5 \cdot 10^{-4}\mathrm{s}^{-1}$, $T_0 = 10^{-4}\mathrm{m}^2/\mathrm{s}^2$, $\nu_0 = 0.01$, and $d = 1\mathrm{m}$, which are close to the probable values of Saturn's A ring, we obtain fairly similar temperature and density profiles for both approaches. These are plotted for different values of the restitution in Fig. 2. The density profiles are strongly peaked at $z = 0$, decaying quickly for increasing height z over the midplane of the ring, while the temperature shows a minimum in the midplane. Furthermore, the profiles get narrower when the interparticle collisions are more dissipative, i.e. the ring becomes thinner.

In spite of the differences of the balance laws that we have pointed out above, both approaches yield similar results. Neither the different functional dependence on temperature and density of the viscous heating term, nor the different restitution dependence of b or β, respectively, nor the additional parameter $\lambda(\epsilon)$ in Eq. (11), alter the vertical stratification near the midplane of the ring considerably. Far off the ring plane however, where the density approaches rapidly zero, we observe a different behavior of the temperature profiles. The temperature of the triaxial kinetic approach finally shows a linear dependence on z. Since the heating is proportional to the density, it gets quite ineffective for large z. Thus, the system has to build up comparatively high temperatures in order to establish energy balance. In contrast the hydrodynamic viscous heating, independent of the density, increases with temperature as z increases. This implies, that lower temperatures are needed in order to achieve energy balance. Finally, the temperature reaches a maximum value and then decreases. This difference in the temperature profiles is dominant however only at latitudes where the density is essentially zero.

In order to establish a connection to the formalism of Goldreich & Tremaine (1978) we integrate the energy balance equations in Eq. (7) and (11) over z. The additional requirement, that the z-integrated heat flow should vanish then leads to the formulae

$$\frac{b(\epsilon)}{c(\epsilon)}\left[\frac{\Omega d}{\nu_0\sqrt{T_0}}\right]^2 = \frac{\int_0^\infty \mathrm{d}z\,\tilde{\nu}^2\sqrt{\tilde{T}}^3}{\int_0^\infty \mathrm{d}z\,\sqrt{\tilde{T}}} \tag{14}$$

$$\frac{\beta(\epsilon)}{\gamma(\epsilon)}\frac{\Omega d}{\nu_0\sqrt{T_0}} = \frac{\int_0^\infty \mathrm{d}z\,\tilde{\nu}^2\sqrt{\tilde{T}}^3}{\int_0^\infty \mathrm{d}z\,\tilde{T}\tilde{\nu}}, \tag{15}$$

respectively, where the definitions $\tilde{\nu} \equiv \nu/\nu_0$ and $\tilde{T} \equiv T/T_0$ are used. By these equations the parameter combinations Ω, d, ν_0, T_0 that correspond to solutions with vanishing z-integrated heat flux are determined, depending on the restitution (Simon & Jenkins, 1994). For instance, we can fix Ω, ν_0 and T_0 and obtain an appropriate d by iterating Eq. (14, 15). This in turn leads via

$$\tau = \frac{3}{d}\int_0^\infty \mathrm{d}z\,\nu \tag{16}$$

to relations between optical depth and restitution. The triaxial ϵ-τ relation reproduces well the simplified formula (18) of Goldreich and Tremaine,and both agree with the hydrodynamic relation in the elastic limit (Fig. 3). The deviations of the hydrodynamic curve are a consequence of the restriction of the validity of the granular transport coefficients to the nearly elastic case.

3. Simulations

In this Section, we compare the theoretical profiles from the numerical solution of Eq. (7, 11) to the results of 3-D event driven N-body simulations, using the code described in (Salo, 1992). Depending on the simulation parameters, we simulate about 500 to 4000 identical particles in a box with periodic boundary conditions. The particles have a diameter of $d = 2\mathrm{m}$, and the simulation box is located at a distance of $r = 10^8\mathrm{m}$ of the central planet with $\Omega = 1.95 \cdot 10^{-4}\mathrm{s}^{-1}$, corresponding to typical values of the optical thick rings of Saturn. Initially, the particles are randomly placed in the box having Keplerian velocities $\boldsymbol{u} = r\Omega\boldsymbol{e}_{\boldsymbol{\varphi}}$ and a superimposed random velocity that corresponds to an initial temperature. In a collision the particles loose energy determined by the coefficient of restitution, while in between two collisions they move on Keplerian trajectories, according to the equations of motion

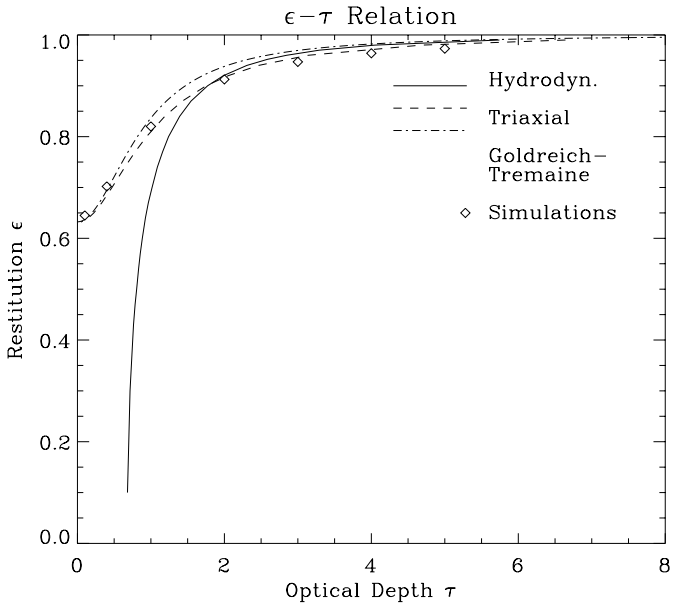

Fig. 3. $\epsilon-\tau$ relations from Eq. (14) and (15) compared to the Goldreich-Tremaine formula Eq. (18). The diamonds are the values taken from the simulations with a mean velocity dispersion of 25 Ωd. The actual critical ϵ values corresponding to an infinite calculation box extent would be slightly larger (about 0.01 at most).

$$\begin{aligned} \ddot{x} - 2\Omega\dot{y} &= 0 \\ \ddot{y} + 2\Omega\dot{x} - 3\Omega^2 y &= 0 \\ \ddot{z} + \Omega^2 z &= 0 \end{aligned} \tag{17}$$

where x and y are the azimuthal and radial coordinates in the co-rotating box, with $x, y, z \ll r$. In order to obtain results comparable to the theoretical investigations of Sect. 2 we carry out the simulations for a constant restitution, and neglect rotational degrees of freedom of the particles.

According to the simplified ϵ-τ formula of Goldreich & Tremaine (1978) for dilute ring systems

$$\left(1-\epsilon^2\right)\left(1+\tau^2\right) \simeq 0.6 \tag{18}$$

for a given value of the constant restitution, equilibrium exists only for one particular value of the optical depth. Practically, in the simulations we fix the optical depth via box size, particle diameter and particle number. Then we adjust the coefficient of restitution so that a finally constant temperature evolves.

If we take for a given τ an ϵ that is too high, the simulated system is unstable since it constantly heats up (see Fig. 4). If ϵ is too low the system cools down monotonically, until an equilibrium is reached that is established by nonlocal transport processes, which are not considered in this paper.

In principle, we could vary the coefficient of restitution in simulations and try to find ϵ-τ pairs that are as close as possible to the limit of stability. However, this poses some practical difficulties. Firstly, as we approach very close to the critical ϵ, the equilibrium temperature rises strongly, and at some point the validity of the local simulation method becomes questionable as the implied mean free path between impacts becomes comparable to the extent of the calculation region. Secondly, our success

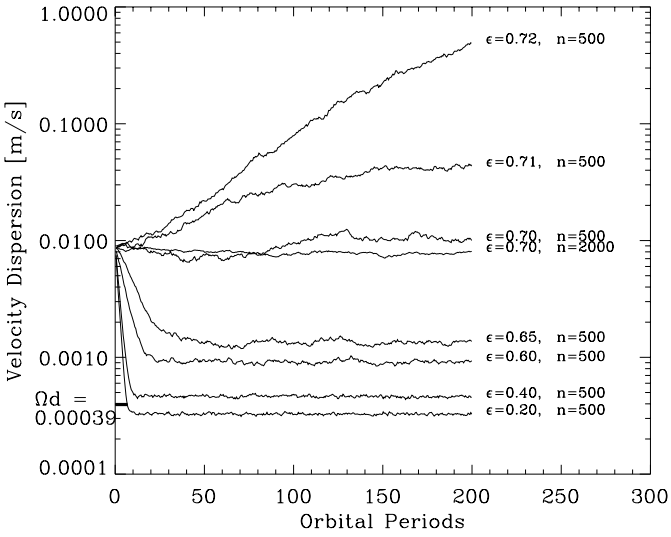

Fig. 4. Development of the scalar velocity dispersion in simulations plotted against the number of orbital periods for various values of the coefficient of restitution. The optical depth is fixed (here $\tau = 0.4$) and the restitution is varied.

in finding critical ϵ-τ pairs would vary in a spurious way with different τ, and the so obtained vertical profiles would correspond to different temperatures, making their direct comparison difficult. For these reasons we chose the strategy to search for ϵ-τ pairs which yield a fixed temperature, corresponding to a velocity dispersion of about 25 Ωd. This rather high temperature assures that the influence of non-local heating, an effect that is not considered in the theoretical formulas, is small and most importantly, stays on a fixed level for all the studied optical depths τ. It also assures that the profiles we obtain are not affected by the size of the calculation region (i.e. by the number of particles). The values of ϵ we obtain for $\tau = 0.1, 0.4, 1.0, 2.0, 3.0, 4.0,$ and 5.0 are $0.645, 0.702, 0.820, 0.9125, 0.947, 0.964,$ and 0.973, respectively. These ϵ-τ pairs fit well with the theoretical relations given in the previous Section (see Fig. 3). In this way we get results that can be compared to the particular solutions of Eq. (7, 11) that fulfill the additional requirements (14) or (15), respectively. The difference as compared to an critical ϵ that corresponded to an infinite calculation region can be expected to be fairly small: For example, for $\tau = 1.0$ additional simulations showed that $\epsilon = 0.830$ is definitely unstable. A choice of these runs is shown in Fig. 5.

We obtain the vertical stratification in the simulation box, by dividing it into layers parallel to the midplane, and then determining the particle number and velocity ellipsoid within each layer. The shown scalar temperature is defined as $T \equiv \frac{1}{3}\mathrm{Tr}\langle(\mathrm{v}_\alpha - \mathrm{u}_\alpha)(\mathrm{v}_\beta - \mathrm{u}_\beta)\rangle$, and the equilibrium velocity dispersion we observe is anisotropic (see Fig. 6).

For nearly elastic collisions ($\epsilon \geq 0.8$), we find for both theoretical ansatzes good agreement with the simulated density profiles, while the agreement of the temperature profiles is not bad, at least near the ring plane. When the collisions get more inelastic ($\epsilon < 0.7$), the hydrodynamic equations overestimate the ring thickness and finally fail to produce even qualitatively

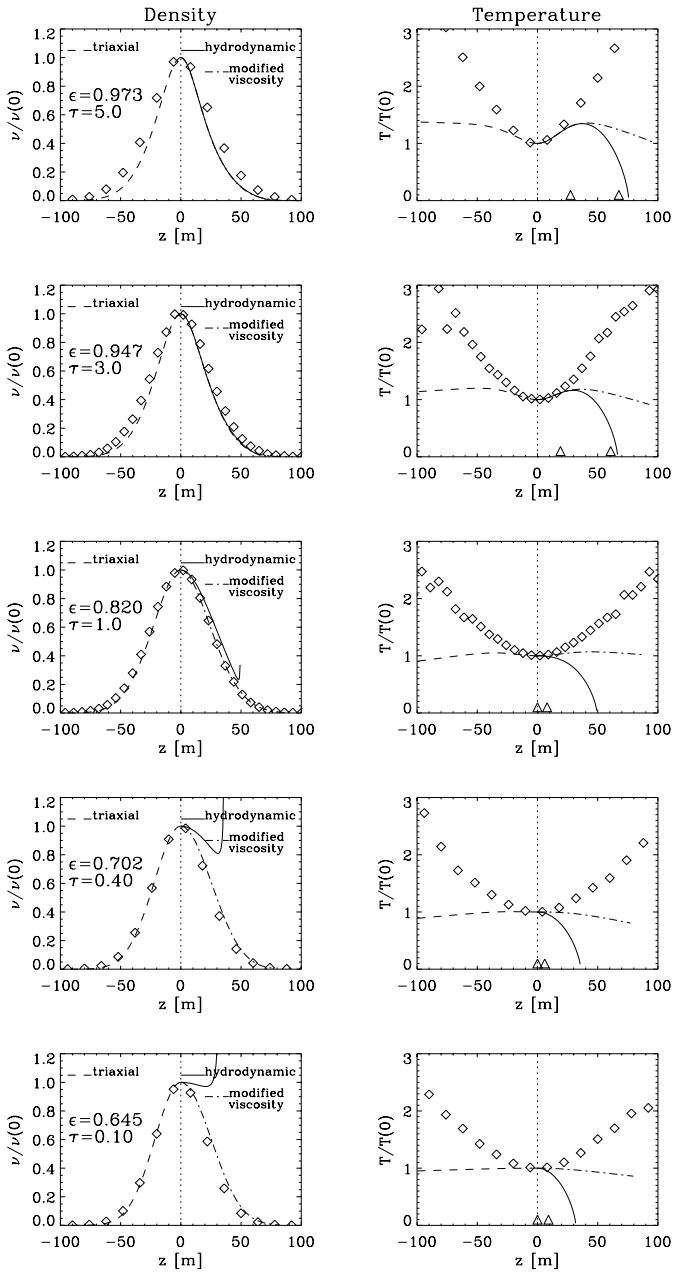

Fig. 5. Density and Temperature profiles from simulations (diamonds) for several values of the coefficient of restitution ϵ and optical depth τ. Theoretical solutions of the hydrodynamic (solid, $z > 0$) and triaxial (dashed, $z < 0$) equations are overplotted. Also shown is a solution of the hydrodynamic equations with a modification of viscosity (dash-dotted $z > 0$), as given by Eq. (13). For these curves the marks on the positive z-axis of the right-hand side frames denote the points where Eq. (13) has actually been applied and where inequality (10) gets false, respectively. For $\tau = 5.0$ and 4.0 the density profile of the modified hydrodynamic treatment is indistinguishable from the hydrodynamic one.

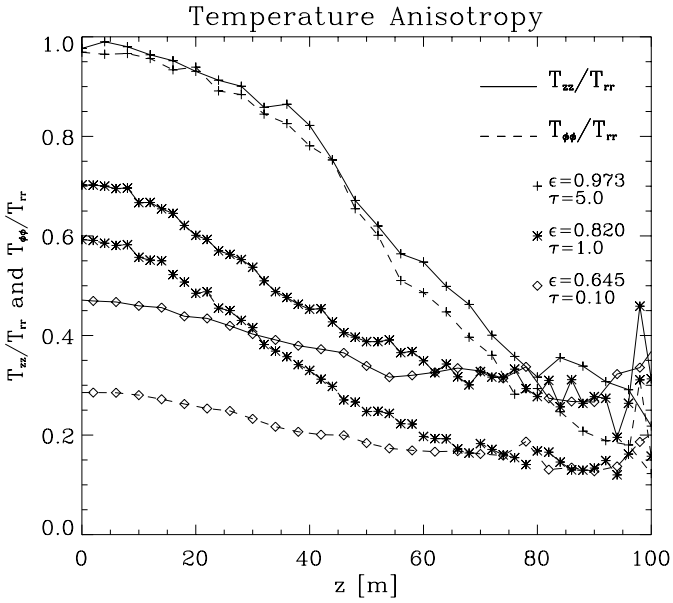

Fig. 6. Z-dependence of the temperature anisotropy of the simulated data for several values of the restitution and optical depth. Plotted are the ratios of vertical to radial and azimuthal to radial component of the temperature tensor, respectively.

correct results. Here, the transport coefficients cease to give a satisfactory description of the transport processes in the ring. However, if we replace in (7) the viscosity η by η^* given by Eq. (13), in the outer layers of the ring where condition (10) is invalid, we obtain density profiles that are in good agreement with the simulations for all values of the restitution we investigated. In the inelastic case the kinetic balance laws lead to a slightly thinner ring as the simulations but the agreement remains good.

Both ansatzes fail to model the simulated temperature profiles in the deeply inelastic case. The theoretical vertical energy balance is achieved for too low temperatures. The reason for this might be in both cases an overestimation of viscous heating. In the hydrodynamic description this is a consequence of the restriction of the viscosity (as given by Eq. (5)) to the nearly elastic case. This implies that for lower ϵ the viscosity in (7) may have to be corrected to smaller values. The triaxial balance law, on the other hand, is based on the assumption of a z-independent temperature anisotropy Simon & Jenkins (1994). However, in the simulations we find that for larger z the radial component of the velocity dispersion becomes more and more dominant, see Fig. 6. Thus, in contrast to the simulations the theoretical heating contribution $\propto T$ to the triaxial energy balance is overestimated by this assumption.

4. Discussion

In the present work we study the vertical stratification of a planetary ring in the framework of a hydrodynamic description of the granular particle ensemble that constitutes the ring. As a consequence of the balance between energy input, driven by the planet's gravity and the cooling due to inelastic collisions, a planetary ring is characterized by a non-uniform distribution of the stationary temperature and density profile. In our approach we assume that this system can be described locally in terms of an isotropic Gaussian phase space distribution plus correc-

tions that are linear in the thermodynamic forces, i.e. the spatial derivatives of density, mean velocity, and temperature, that account for the non-equilibrium state of the system. This treatment allows a consistent description of the fluxes of mass, momentum and energy in terms of transport coefficients that are related linearly to the corresponding thermodynamic forces. We investigate the vertical components of the associated hydrodynamic balance laws and solve them numerically.

This approach is compared to the one of Simon & Jenkins (1994), who investigated the vertical ring structure in terms of a triaxial Gaussian velocity distribution. This treatment is on the one hand more general, using a non-isotropic velocity ellipsoid, on the other hand no further modification of the distribution function due to the non-equilibrium state is considered. A heat flux contribution that is derived under more general considerations by Zhang (1993) is added by hand. Provided we apply a simple correction to the granular viscosity stemming from the restriction of the mean free path of a ring particle by the epicyclic motion, both approaches yield fairly similar solutions. The theoretical density profiles are in a very good agreement with the results of numerical simulations we carried out for different values of the optical depth and coefficient of restitution. We find that the simulated ϵ-τ values obey the same relation as the triaxial Gaussian approach and as the linear hydrodynamic approach in the elastic limit, when the condition of a vanishing z-integrated heat flux is superimposed on the theoretical balance laws.

Further, the simulations show that the ratios of the principal axes of the velocity ellipsoid as well as its orientation are depending on latitude. This dependence has been neglected in the derivation of the triaxial balance equations Simon & Jenkins (1994) for reasons of simplicity. Our findings however point at the possibility that it might become important in the description of vertical energy balance in a planetary ring.

The theoretical models as presented in Sect. 2 are based on the assumption that the dissipation of energy in the interparticle collisions can be described in terms of a velocity independent coefficient of restitution. However, we know from experiments (Raman, 1918; Hatzes et al., 1988; Supulver et al., 1995) as well as from theoretical considerations using various different ansatzes (Pöschl, 1928; Andrews, 1930; Pao, 1955; Kuwabara & Kono, 1987; Spahn et al., 1995; Morgado & Oppenheim, 1997; Thornton, 1997), that in general the restitution is a function of the impact velocity. As pointed out in the considerations of Sect. 3, the velocity independent restitution does not lead to a model of a planetary ring that has a stable equilibrium state. This drawback is removed when the coefficient of restitution becomes a monotonically decreasing function of the impact velocity. The corresponding extensions of the kinetic theory of granular matter, in particular the calculation of the necessary corrections to the distribution function and the transport coefficients, are subject of our ongoing and future work.

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