Discontinuous Petrov-Galerkin finite element method for numerical analysis of partial differential equations

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DPG-FEM: Method for all seasons!?

- Discovered by Demkowicz and Gopalakrishnan in 2009
- Promise: Utilize *hp*-adaptive FEM for any (non-)linear PDE
- Main idea: compute test functions **on the fly** to guarantee numerical stability **automatically**

Collaborators:

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Outline

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2. Petrov-Galerkin methods for variational problems
3. Diffusion problem: DPG-FEM vs. LS-FEM
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Petrov-Galerkin method for linear systems

- Consider

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times n}$$

and let $U_m, V_m \subset \mathbb{R}^n$ be subspaces of dimension m

- **Petrov-Galerkin method**: Find $\mathbf{u} \in U_m$ such that

$$\mathbf{v}^T(\mathbf{b} - \mathbf{Au}) = 0 \quad \forall \mathbf{v} \in V_m$$

- The choice $V_m = \mathbf{A}U_m$ minimizes the residual $||\mathbf{b} - \mathbf{Au}||$:

$$||\mathbf{b} - \mathbf{A}(\mathbf{u} + \mathbf{d})||^2 = ||\mathbf{b} - \mathbf{Au}||^2 - 2(\mathbf{Ad})^T(\mathbf{b} - \mathbf{Au}) + ||\mathbf{Ad}||^2$$

- Alternatively one may employ **Bubnov-Galerkin method** with $V_m = U_m$ to the normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$
- Practical algorithms minimize the residual in the Krylov subspace

$$U_m = \mathrm{span}\{\mathbf{b}, \mathbf{Ab}, \mathbf{A}^2\mathbf{b}, \ldots, \mathbf{A}^{m-1}\mathbf{b}\}$$

in an elegant way (Arnoldi method)

Petrov-Galerkin method for variational problems

- **Variational problem**: Find $u \in U$ such that

$$\mathcal{B}(u, v) = \ell(v) \quad \forall v \in V,$$

  where U, V are (real) Hilbert spaces

- **Petrov-Galerkin approximation**: Find $u_h \in U_h \subset U$ such that

$$\mathcal{B}(u_h, v) = \ell(v) \quad \forall v \in V_h$$

- Let the **trial space** be $U_h = \mathrm{span}\{e_1, \ldots, e_n\}$ and define the **test space** as $V_h = T(U_h)$, where $T : U \to V$ is defined as

$$(Tu, v)_V = \mathcal{B}(u, v) \quad \forall v \in V$$

- The ideal PG method **minimizes the residual**

$$|||u - u_h|||_U \doteq ||T(u - u_h)||_V = \sup_{v \in V} \frac{\mathcal{B}(u - u_h, v)}{||v||_V}$$

Petrov-Galerkin method for variational problems (cont.)

Theorem

If the variational formulation is wellposed, that is,

$$1^\circ \quad \{w \in U \;:\; \mathcal{B}(w,v) = 0 \quad \forall v \in V\} = \{0\}$$

$$2^\circ \quad \alpha ||v||_V \le \sup_{w \in U} \frac{\mathcal{B}(w,v)}{||w||_U} \le C||v||_V \quad \forall v \in V$$

and an **approximation** $T^r : U \to V^r$ *is* **computed** *such that*

$$3^\circ \quad \exists\, \Pi : V \to V^r \text{ s.t. } ||\Pi|| \le c \;\&\; \mathcal{B}(w_h, v - \Pi v) = 0 \quad \forall w_h \in U_h,\; v \in V$$

then it holds

$$||u - u_h||_U \le \frac{Cc}{\alpha} \min_{w_h \in U_h} ||u - w_h||_U$$

Diffusion problem: DPG formulation

- **Poisson's equation** in $\mathbb{R}^2$:

$$-\nabla^2 u = f \quad \text{in } \Omega \quad \& \quad u = 0 \quad \text{on } \partial\Omega.$$

- Variational form: Find $(u, \hat{q}_n) \in H_0^1(\Omega) \times H^{-1/2}(\partial\Omega_h)$ s.t.

$$(\nabla u, \nabla v)_{\Omega_h} - \langle \hat{q}_n, v \rangle_{\partial\Omega_h} = (f, v)_\Omega \quad \forall v \in H^1(\Omega_h)$$

- Here Ω_h is a **mesh** of Ω and for regular functions f, g

$$(f, g)_{\Omega_h} = \sum_{K \in \Omega_h} \int_K fg \,\mathrm{d}x \quad \& \quad \langle f, g \rangle_{\partial\Omega_h} = \sum_{K \in \Omega_h} \int_{\partial K} fg \,\mathrm{d}s$$

- The **non-standard Sobolev spaces** are:

$$H^1(\Omega_h) = \{v \in L_2(\Omega) \ : \ v|_K \in H^1(K) \ \forall K \in \Omega_h\}$$
$$H^{-1/2}(\partial\Omega_h) = \{\eta \ : \ \exists q \in H(\text{div}, \Omega) \text{ s.t. } \eta|_{\partial K} = q \cdot n|_{\partial K}\}$$

Diffusion problem: DPG formulation (cont.)

- The norms are:

$$||(w, \hat{r}_n)||_U^2 = ||\nabla w||_{L_2(\Omega)}^2 + ||\hat{r}_n||_{H^{-1/2}(\partial\Omega_h)}^2$$
$$||v||_V^2 = (\nabla v, \nabla v)_{\Omega_h} + (v, v)_{\Omega_h}$$

where the space of numerical fluxes is normed as

$$||\hat{r}_n||_{H^{-1/2}(\partial\Omega_h)} = \inf\{||q||_{H(\text{div},\Omega)} : q \in H(\text{div},\Omega) \text{ s.t. } \hat{r}_n|_{\partial K} = q\cdot n|_{\partial K}\}$$

Theorem

The primal DPG formulation of the diffusion problem is wellposed with ***mesh-independent*** *stability constants C and α.*

Proof.

See [DG13]. □

Diffusion problem: DPG approximation

- Assume that Ω_h is a **shape-regular** partitioning of Ω into **convex quadrilaterals**
- **Conforming** DPG-FE trial space of degree k and test space of degree r:

$$\begin{aligned} U_h &= \{(w, \hat{r}_n) \in U \ : \ w|_K \in Q_k(K),\ \hat{r}_n \in P_{k-1}(\partial K)\} \\ V^r &= \{v \in V \ : \ v|_K \in Q_r(K)\} \end{aligned}$$

where

$$\begin{aligned} Q_k(K) &= \{w \in L_2(K) \ : \ w = \hat{w} \circ F_K^{-1}, \hat{w} \in P_{k,k}(\hat{K})\} \\ P_k(\partial K) &= \{\hat{r}_n \in L_2(\partial K) \ : \ \hat{r}_n|_E \in P_k(E) \text{ for each edge } E \text{ on } \partial K\} \end{aligned}$$

- $F_K : \hat{K} \to \mathbb{R}^2$ is the **bilinear mapping** onto $K = F_K(\hat{K})$

Diffusion problem: error estimate

Theorem

Let $(u_h, \hat{q}_{n,h})$ be the DPG approximation of degree k to the diffusion problem with $r = k + 2$. Then

$$||u - u_h||_{H^1(\Omega)} + ||\hat{q}_n - \hat{q}_{n,h}||_{H^{-1/2}(\partial\Omega_h)} \leq C \min_{(w_h, \hat{r}_{n,h}) \in U_h} \left(||u - w_h||_{H^1(\Omega)} + ||\hat{q}_n - \hat{r}_{n,h}||_{H^{-1/2}(\partial\Omega_h)} \right)$$

Proof.

There exists a bounded projector $\Pi_{k+2} : H^1(K) \to Q_{k+2}(K)$ such that

$$\int_K (\Pi_{k+2} v - v) w_k \, \mathrm{d}x = 0 \quad \forall w_k \in Q_k(K),$$

$$\int_{\partial K} (\Pi_{k+2} v - v) \mu_k \, \mathrm{d}s = 0 \quad \forall \mu_k \in P_k(\partial K),$$

see [CCN14]. □

Diffusion problem: convergence rates

Theorem

Let $(u_h, \hat{q}_{n,h})$ be the DPG approximation of degree k to the diffusion problem with $r = k + 2$. Then

$$||u - u_h||_{H^1(\Omega)} + ||\hat{q}_n - \hat{q}_{n,h}||_{H^{-1/2}(\partial\Omega_h)} \leq Ch^k \left(|u|_{H^{k+1}(\Omega)} + |\operatorname{div} q|_{H^k(\Omega)} \right)$$

Proof.

The first term is bounded by standard approximation theory. The second term can be bounded by using $H(\text{div})$-projection of the flux $q = -\nabla u$ on the space

$$ABF_{k-1}(\hat{K}) = P_{k+1,k-1}(\hat{K}) \times P_{k-1,k+1}(\hat{K})$$

since the normal components are polynomials of degree $k - 1$. □

Diffusion problem: comparison with LS-FEM

- Least squares FEM for the first order system minimizes

$$\mathcal{F}(w,r) = ||r + \nabla w||^2_{L_2(\Omega)} + ||\operatorname{div} r - f||^2_{L_2(\Omega)}$$

  over a subspace $W_h \times R_h \subset H^1(\Omega) \times H(\operatorname{div},\Omega)$
- For any choice of the subspace

$$\begin{aligned} &||u - u_h||_{H^1(\Omega)} + ||q - q_h||_{H(\operatorname{div},\Omega)} \\ &\qquad \le C\left(\min_{w_h \in W_h} ||u - w_h||_{H^1(\Omega)} + \min_{r_h \in R_h} ||q - r_h||_{H(\operatorname{div},\Omega)}\right) \end{aligned}$$

- If R_h is based on Raviart-Thomas space

$$RT_{k-1}(\hat{K}) = P_{k,k-1}(\hat{K}) \times P_{k-1,k}(\hat{K})$$

  the best known projection estimate is

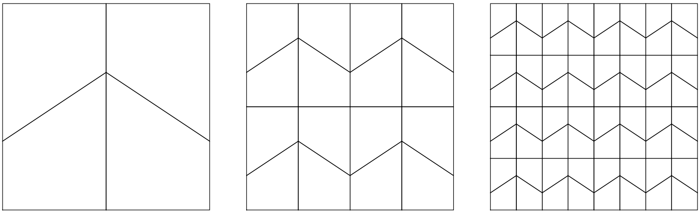
$$||\operatorname{div}(q - \Pi_k q)||_{L_2(\Omega)} \le Ch^{k-1}|\operatorname{div} q|_{H^k(\Omega)}$$

Diffusion problem: manufactured solution

- Consider Poisson's equation in $\Omega = (0,1)^2$ with the solution

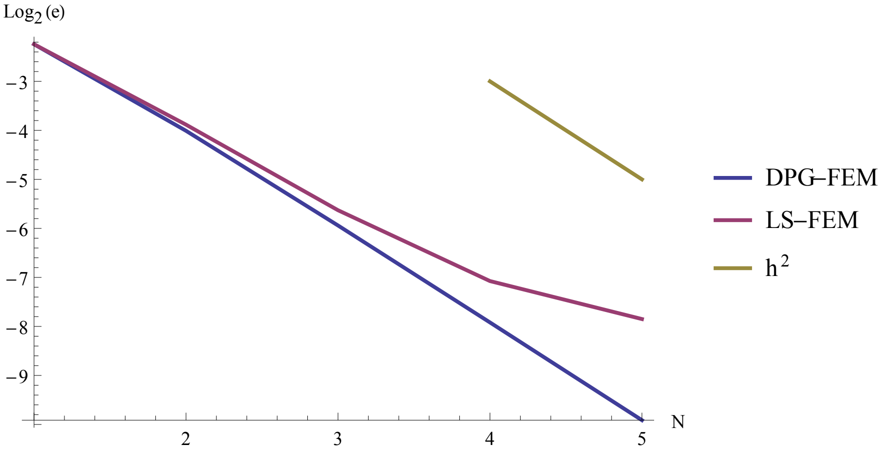
$$u(x_1, x_2) = \cos(\pi x_1)\cos(\pi x_2)$$

- Lowest-order DPG-FEM vs. LS-FEM on trapezoidal $2^N \times 2^N$ -meshes, $N = 1, 2, 3, \ldots$



Diffusion problem: DPG-FEM vs. LS-FEM

Error $e = ||u - u_h||_{L_2(\Omega)}$, when $N = 1, 2, 3, 4, 5$:



Summary and remarks

- DPG-FEM is **automatically numerically stable**
- DPG-FEM has a **built-in local error evaluator**
- The computational cost of DPG-FEM is comparable to mixed finite methods
- The **robustness** of the method can be improved by changing the test space inner product

References

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