Aalto University

Scordelis-Lo Roof Revisited

Antti H. Niemi
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Shell Finite Elements

- **The error estimates for *standard finite elements* for shells are of the form**

$$\frac{|||\mathbf{u}-\mathbf{u}_{h,p}|||}{|||\mathbf{u}|||} \leq C\frac{L}{t}\left(\frac{h}{L}\right)^p$$

- **There are formulations (like MITC-type) for which**

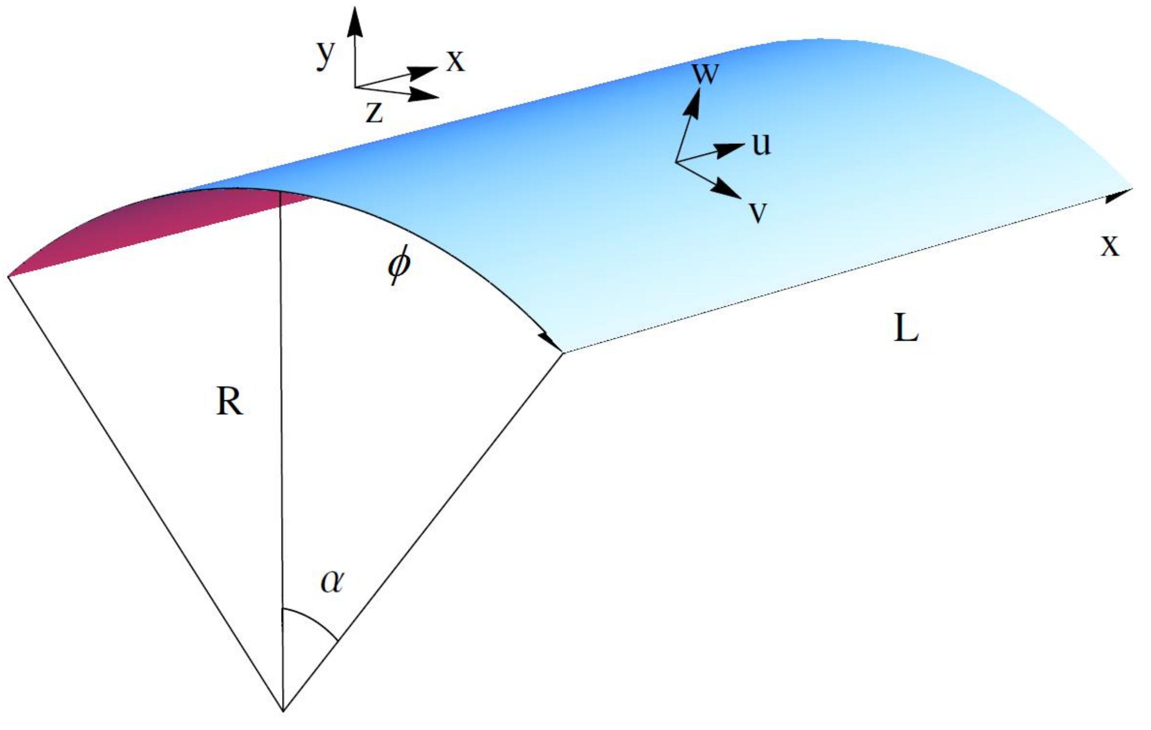
$$\frac{|||\mathbf{u}-\mathbf{u}_{h,p}|||_h}{|||\mathbf{u}|||} \leq C\left(\frac{h}{L}\right)^p$$

***under very specific geometric assumptions*.**

- **Practical shell finite element formulations are validated using *benchmark computations***

Scordelis-Lo Roof Problem Setup

Problem Setup (SI-units)



$$R = 25\,\mathrm{m}$$
$$L = 50\,\mathrm{m}$$
$$\alpha = 40^\circ$$

Problem Setup (cont.) (SI-units)

- **Material:**
  - Isotropic material with Young's modulus set to $E = 4.32 \cdot 10^8\,\mathrm{N/m^2}$.
  - Poisson's ratio set to $\nu = 0.0$.
- **Kinematic constraints:**
  - The outer curved edges are constrained against vertical translations (*diaphragm support*).
  - The outer straight edges are *free*.
- **Load:**
  - A force per are unit of $-90\,\mathrm{N/m^2}$ in the vertical direction is applied on the surface.

Benchmark Setting

- **Set the thickness to $t = 0.25\,\mathrm{m}$ and determine the value of the vertical displacement at the center of the free edge.**
- **The reference solution quoted in the literature for the midside vertical displacement is −0.309 m.**

Scordelis, A. C., & Lo, K. S. (1964). Computer Analysis of Cylindrical Shells. *ACI Journal Proceedings*, *61*(5). doi:10.14359/7796

Classical Shell Model (Love-Novozhilov-Koiter)

- **Total potential energy of the shell is**

$$\mathcal{F}(\boldsymbol{u}) = \frac{Et}{2}\int_\omega (\beta_{11}^2 + 2\beta_{12}^2 + \beta_{22}^2)\,\mathrm{d}x\mathrm{d}\phi + \frac{Et^3}{24}\int_\omega (\kappa_{11}^2 + 2\kappa_{12}^2 + \kappa_{22}^2)\,\mathrm{d}x\mathrm{d}\phi$$
$$- p\int_\omega (\sin\phi\cdot v - \cos\phi\cdot w)\,\mathrm{d}x\mathrm{d}\phi,$$

where $\boldsymbol{u} = (u, v, w)$ **and** $\omega = (0, L)\times(-\alpha,\alpha)$**.**

- **The membrane strains are defined as**

$$\beta_{11} = \frac{\partial u}{\partial x},\quad \beta_{22} = \frac{1}{R}\left(\frac{\partial v}{\partial \phi} + w\right),\quad \beta_{12} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right)$$

Classical Shell Model (cont.) Bending Strains

- **Shallow shell theory (LNK-SS):**

$$\kappa_{11} = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_{22} = \frac{1}{R^2}\frac{\partial w}{\partial \phi^2}, \quad \kappa_{12} = \frac{1}{R}\frac{\partial^2 w}{\partial x \partial \phi}.$$

- **Novozhilov's model (LNK-S):**

$$\kappa_{11} = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_{22} = \frac{1}{R^2}\frac{\partial w}{\partial \phi^2} - \frac{1}{R^2}\frac{\partial v}{\partial \phi}, \quad \kappa_{12} = \frac{1}{R}\frac{\partial^2 w}{\partial x \partial \phi} - \frac{1}{R}\frac{\partial v}{\partial x}.$$

- **Vlasov's model (Vlasov):**

$$\kappa_{11} = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_{22} = \frac{1}{R^2}\frac{\partial w}{\partial \phi^2} + \frac{w}{R^2}, \quad \kappa_{12} = \frac{1}{R}\frac{\partial^2 w}{\partial x \partial \phi} - \frac{1}{2R}\frac{\partial v}{\partial x} + \frac{1}{2R^2}\frac{\partial u}{\partial \phi}.$$

Semi-analytic Solution

Ansatz

- **Kinematic constraints at $x = 0, L$ are satisfied for**

$$u = \sum_{m=1}^{\infty} U_m(\phi)\cos(\beta_m x),\; v = \sum_{m=1}^{\infty} V_m(\phi)\sin(\beta_m x),\; w = \sum_{m=1}^{\infty} W_m(\phi)\sin(\beta_m x),$$

where

$$\beta_m = \frac{m\pi}{L}.$$

- **The constant load can be represented as**

$$p = \sum_{m=1}^{\infty} P_m \sin(\beta_m x), \quad P_m = \begin{cases} \frac{4p}{\pi m}, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}$$

Euler Equations
Shallow Shell Theory

- **Substitution of the Ansatz in $\mathcal{F}(u)$ and minimization the energy with respect to U_m, V_m, W_m gives**

$$\beta_m^2 U_m - \frac{\beta_m}{2R} V_m' - \frac{U_m''}{2R^2} = 0,$$

$$\frac{\beta_m^2}{2} V_m + \frac{\beta_m}{2R} U_m' - \frac{1}{R^2}(V_m'' - W_m') = \frac{P_m}{Et} \sin\phi,$$

$$\frac{1}{R^2}(V_m' + W) + \frac{t^2}{12}\left(\frac{W_m^{(4)}}{R^4} - \frac{\beta_m^2}{2R^2} W_m'' + \beta_m^4 W_m\right) = -\frac{P_m}{Et}\cos\phi.$$

- **These constitute a system of *constant-coefficient* ODEs of total degree *eight,* valid when $\phi \in (-\alpha, \alpha)$.**

Natural Boundary Conditions Shallow Shell Theory

- **The natural boundary conditions at the free edges at $\phi = \pm\alpha$ are**

$$\frac{1}{R}U_m' + \beta_m V_m = 0,$$
$$V_m' + W_m = 0,$$
$$\beta_m^2 W_m' - \frac{1}{2R^2}W_m^{(3)} = 0,$$
$$W_m'' = 0.$$

- **These correspond to the conditions of vanishing normal forces, effective shear force and bending moment.**

Solution

- **The solution of the system can be expressed as**

$$(U_m, V_m, W_m) = (\hat{U}_m, \hat{U}_m, \hat{U}_m) + \sum_{i=1}^{8} C_i \boldsymbol{v}_i e^{\lambda_i \phi},$$

where

  - $(\hat{U}_m, \hat{U}_m, \hat{U}_m)$ is a special solution of the *non-homogeneous* system.
  - $(\lambda_i, \boldsymbol{v}_i)$ are the eigenpairs associated to the *homogeneous* system.

- $(\hat{U}_m, \hat{U}_m, \hat{U}_m)$ **is found easily in the form**

$$(\hat{U}_m(\phi), \hat{U}_m(\phi), \hat{U}_m(\phi)) = (\tilde{U}_m \cos\phi, \tilde{V}_m \sin\phi, \tilde{W}_m \cos\phi)$$

Solution (cont.)

- The special solution and the eigenvalues can be determined by symbolic computation for each m.
- The eigenvectors are solved numerically.
- The most difficult part is the determination of the integration coefficients $C_i,\ i = 1, \ldots, 8$.
- These can be solved with Mathematica using arbitrary high numerical precision.

Numerical Results

Midpoint Vertical Displacement

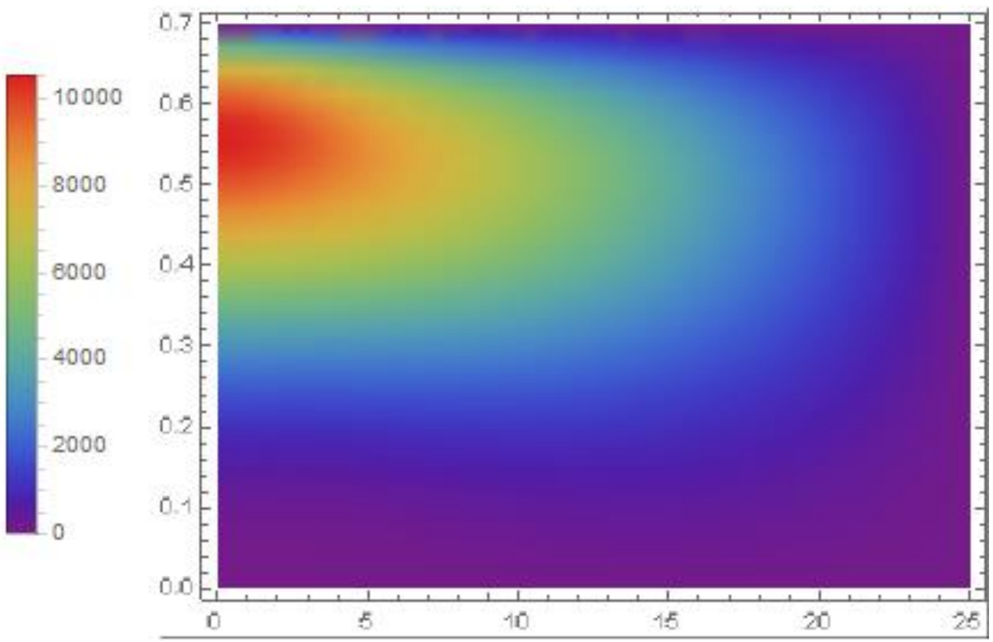
- **Original problem setup with t=0.25 m (R/t=100):**

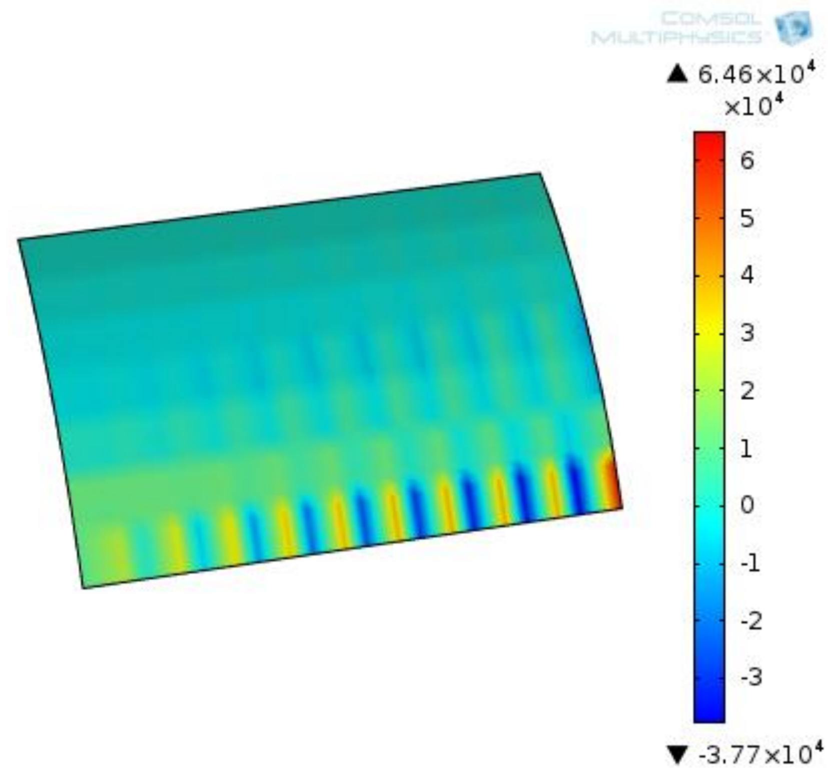
LNK-SS	LNK-S	Vlasov	COMSOL	MITC4S
-0.309 m	-0.299 m	-0.312 m	-0.302 m	-0.302 m

- **Academic problem setup with t=0.025 m (R/t=1000):**

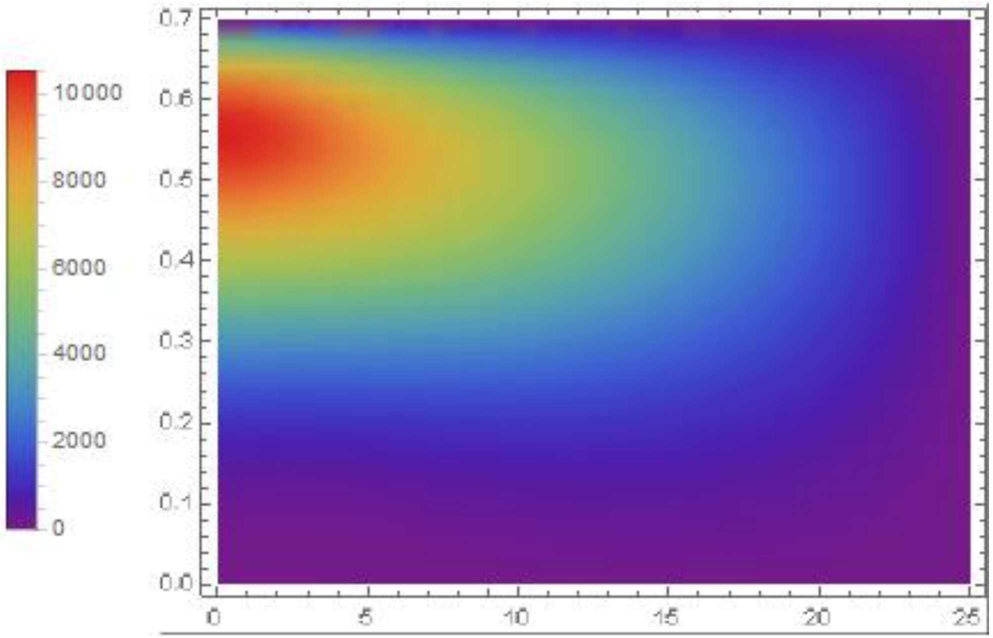
LNK-SS	LNK-S	Vlasov	COMSOL	MITC4S
-30.9 m	-32.0 m	-32.0 m	-32.0 m	-32.0 m

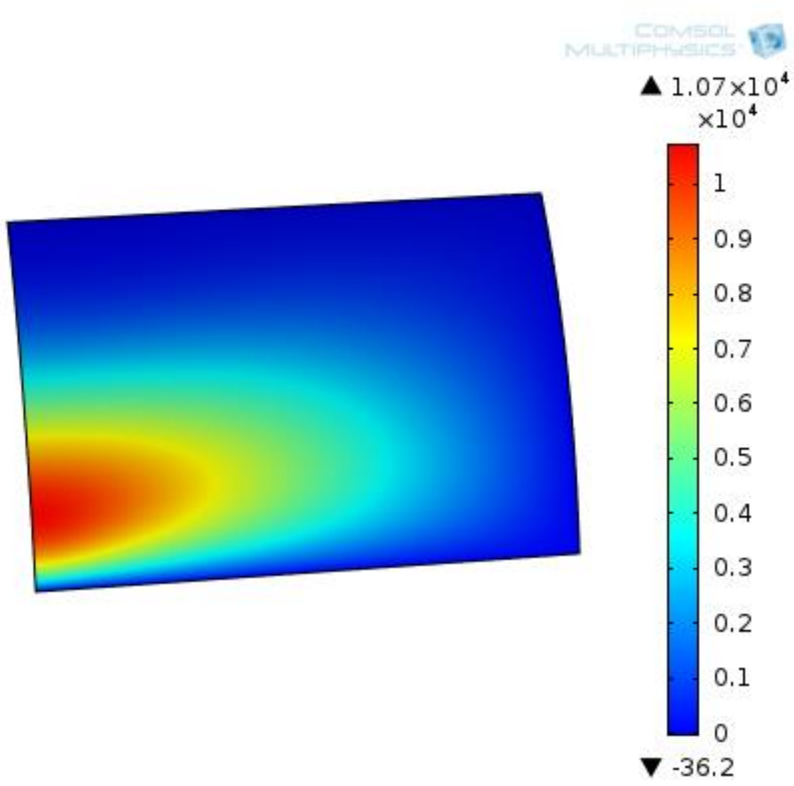
In-plane Shear Force
LNK-S vs. COMSOL (4-node, normal)





In-plane Shear Force
LNK-S vs. COMSOL (4-node, extra-fine)





Concluding Remarks

- **We have presented a semi-analytic solution technique for the classical Scordelis-Lo roof model problem.**

- **Our approach allows to study different variants of shell models for different values of the thickness.**

- **The solution technique can be used to benchmark shell finite elements but also directly in the design of cylindrical shell roofs!**