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A family of triangular shell elements

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Shell finite elements

- **In the finite element modelling of shell structures parametric error amplification, or locking, is detected for various shell deformation types.**

- **The numerical phenomenon is especially harmful for the lowest-order (*p*=1) finite element approximation and significant mesh over-refinement is sometimes needed.**

- **A long-standing approach is the derivation of special low-order formulations that avoid the parametric error growth.**

Shell Finite Elements

- The error estimates for *standard finite elements* for shells are of the form

$$\frac{|||\mathbf{u}-\mathbf{u}_{h,p}|||}{|||\mathbf{u}|||} \leq C\frac{L}{t}\left(\frac{h}{L}\right)^p$$

- There are special formulations (like quadrilateral MITC elements) for which

$$\frac{|||\mathbf{u}-\mathbf{u}_{h,p}|||_h}{|||\mathbf{u}|||} \leq C\left(\frac{h}{L}\right)^p$$

under very specific geometric assumptions.

- Practical shell finite element formulations are validated using *benchmark computations*

Shell theory for FEM

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Shell coordinates

- **Assume that the shell body has a constant thickness *t* and that its mid-surface is discretized using triangular elements.**

- **The stiffness over each triangle *K* is calculated using shell theory in a *local* curvilinear coordinate system (x, y, ζ), where**
  - $(x, y) \in K$ are some chosen Cartesian coordinates on K
  - $\zeta \in (-t/2, t/2)$ is the coordinate along the unit normal vector $\vec{n}(x, y)$ to the shell mid-surface

- **The (reasonable) meshing assumption is that each *K* is so small that the coordinates can be assumed *orthogonal* on *K*.**

Shell kinematics

- **According to the standard kinematic hypothesis the displacement vector field is assumed to be of the form**

$$\vec{U}(x,y,\zeta) = (u_\lambda(x,y) + \zeta\theta_\lambda(x,y))\vec{e}_\lambda(x,y) + w(x,y)\vec{n}(x,y),$$

  where
  - $\boldsymbol{u} = (u_1, u_2)$ are the tangential displacements of the mid-surface
  - w is the transverse deflection
  - $\boldsymbol{\theta} = (\theta_1, \theta_2)$ are the angles of rotation of the normal
- **The tangential displacements and the rotations follow here the tangential directions $\vec{e}_1$ and $\vec{e}_2$ along the *x*- and *y*-coordinate lines, respectively.**

Curvilinear strains

- **Referring to the curvilinear coordinates (x, y, ζ), the in plane components of the linearized Green-Lagrange strain tensor can be expanded as**

$$e_{\alpha\beta} \approx \varepsilon_{\alpha\beta} + \zeta\kappa_{\alpha\beta}, \quad \alpha, \beta = 1, 2.$$

- **The *membrane strain tensor* can be written as**

$$\varepsilon_{\alpha\beta} \approx \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) - b_{\alpha\beta}w,$$

where

$$b_{\alpha\beta} = -\vec{e}_\alpha \cdot \vec{n}_{,\beta}, \quad \alpha, \beta = 1, 2,$$

are the coefficients of the second fundamental form of the mid-surface.

Curvilinear strains (cont.)

- **Introducing the coefficients of the third fundamental form**

$$c_{\alpha\beta} = \vec{n}_{,\alpha} \cdot \vec{n}_{,\beta}, \quad \alpha, \beta = 1, 2,$$

the elastic curvature tensor comes out as

$$\kappa_{\alpha\beta} \approx \frac{1}{2}(\theta_{\alpha,\beta} + \theta_{\beta,\alpha}) + c_{\alpha\beta} w - \frac{1}{2}(b_{\alpha\lambda} u_{\lambda,\beta} + b_{\beta\lambda} u_{\lambda,\alpha}), \quad \alpha, \beta = 1, 2$$

- **Because** $c_{\alpha\beta} \approx b_{\alpha\lambda} b_{\lambda\beta}$**, it is possible to write**

$$\kappa_{11} \approx \underline{\theta_{1,1} + b_{12}(b_{12} w - u_{2,1})} - b_{11}\varepsilon_{11},$$

$$\kappa_{22} \approx \underline{\theta_{2,2} + b_{12}(b_{12} w - u_{1,2})} - b_{22}\varepsilon_{22},$$

$$\kappa_{12} \approx \underline{\frac{1}{2}(\theta_{1,2} + \theta_{2,1}) + \frac{b_{11}}{2}(b_{12} w - u_{1,2}) + \frac{b_{22}}{2}(b_{12} w - u_{2,1})} - \frac{b_{12}}{2}(\varepsilon_{11} + \varepsilon_{22}).$$

Curvilinear strains (cont.)

- **The transverse shear strains can be written as**

$$\gamma_\alpha = 2e_{\alpha 3} = \theta_\alpha + b_{\alpha\lambda} u_\lambda + w_{,\alpha}, \quad \alpha = 1, 2$$

- **The strain energy functional reads**

$$U_K(\boldsymbol{u}, w, \boldsymbol{\theta}) = \frac{1}{2} \int_K (n_{\alpha\beta}\varepsilon_{\alpha\beta} + q_\alpha \gamma_\alpha + m_{\alpha\beta}\kappa_{\alpha\beta})\, dxdy$$

where, assuming linearly elastic isotropic material,

$$n_{\alpha\beta} = \frac{Et}{1-\nu^2}\left[(1-\nu)\varepsilon_{\alpha\beta} + \nu\varepsilon_{\lambda\lambda}\delta_{\alpha\beta}\right],$$

$$q_\alpha = \frac{Et}{2(1+\nu)}\gamma_\alpha,$$

$$m_{\alpha\beta} = \frac{Et^3}{12(1-\nu^2)}\left[(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{\lambda\lambda}\delta_{\alpha\beta}\right]$$

Finite element formulations

Courant triangle

- **We assume linear approximation for each displacement component separately on each *K*.**
- **The membrane and transverse shear strains must be reduced to circumvent numerical locking.**
- **To introduce the different methods, we denote by $\boldsymbol{F}_K = (x_K, y_K)$ the affine mapping of the reference triangle $\hat{K}$ onto K and define the Jacobian as**

$$\boldsymbol{J}_K = \begin{pmatrix} \frac{\partial x_K}{\partial \hat{x}} & \frac{\partial x_K}{\partial \hat{y}} \\ \frac{\partial y_K}{\partial \hat{x}} & \frac{\partial y_K}{\partial \hat{y}} \end{pmatrix}$$

Strain reductions

- **We define on the reference triangle the FE spaces**

$$\boldsymbol{S}(\hat{K}) = \{\hat{\boldsymbol{s}} = \begin{pmatrix} a + c\hat{y} \\ b + c\hat{x} \end{pmatrix} \ : \ a, b, c \in \mathbb{R}\}$$

$$\boldsymbol{M}(\hat{K}) = \{\hat{\boldsymbol{\tau}} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \ : \ a, b, c \in \mathbb{R}\}$$

- **The associated DOFs are**

$$\hat{\boldsymbol{s}} \mapsto \int_{\hat{e}} \hat{\boldsymbol{s}}^T \hat{\boldsymbol{t}} \, d\hat{s} \text{ for every edge } \hat{e} \text{ of } \hat{K},$$

$$\hat{\boldsymbol{\tau}} \mapsto \int_{\hat{e}} \hat{\boldsymbol{t}}^T \hat{\boldsymbol{\tau}} \hat{\boldsymbol{t}} \, d\hat{s} \text{ for every edge } \hat{e} \text{ of } \hat{K}.$$

Strain reductions (cont.)

- **The corresponding spaces associated to a general *K* are**

$$\boldsymbol{S}(K) = \{\boldsymbol{s} = \boldsymbol{J}_K^{-T}\hat{\boldsymbol{s}} \circ \boldsymbol{F}_K^{-1} = \mathcal{S}_K(\hat{\boldsymbol{s}}) \ : \ \hat{\boldsymbol{s}} \in \boldsymbol{S}(\hat{K})\}$$
$$\boldsymbol{M}(K) = \{\boldsymbol{\tau} = \boldsymbol{J}_K^{-T}(\hat{\boldsymbol{\beta}} \circ \boldsymbol{F}_K^{-1})\boldsymbol{J}_K^{-1} = \mathcal{M}_K(\hat{\boldsymbol{\tau}}) \ : \ \hat{\boldsymbol{\tau}} \in \boldsymbol{M}(\hat{K})\}$$

- **The projectors are defined as**

$$\boldsymbol{\Pi}_K = \mathcal{M}_K \circ \boldsymbol{\Pi}_{\hat{K}} \circ \mathcal{M}_K^{-1} \text{ and } \boldsymbol{\Lambda}_K = \mathcal{S}_K \circ \boldsymbol{\Lambda}_{\hat{K}} \circ \mathcal{S}_K^{-1}$$

where $\boldsymbol{\Lambda}_{\hat{K}} : \boldsymbol{H}^1(\hat{K}) \to \boldsymbol{S}(\hat{K})$ and $\boldsymbol{\Pi}_{\hat{K}} : \boldsymbol{H}^1(\hat{K}) \to \boldsymbol{M}(\hat{K})$ **are well-defined on the reference triangle.**

- **The DOFs (tangential components) are preserved.**

The element family

- **MITC3C:** $\gamma \hookrightarrow \Lambda_K \gamma$
- **MITC3S:** $\varepsilon \hookrightarrow \mathbf{\Pi}_K \varepsilon, \quad \gamma \hookrightarrow \Lambda_K \gamma$
- **Stabilized variants of both elements can be introduced by modifying the shear modulus as**

$$G \hookrightarrow G_K = \frac{t^2}{t^2 + \alpha_K h_K^2} \cdot G.$$

- **Here** α_K **is a positive stabilization parameter independent of** t **and** h_K.

Implementation

Skew coordinate transformations

- **Geometric input data: triangulation of the middle surface and nodal normal vectors.**
- **Two orthogonal directions $\vec{g}_1, \vec{g}_2$ to the nodal normals are generated and a skew coordinate transformation**

$$u_\alpha^K \circ T_K = \tilde{u}_\lambda \vec{g}_\lambda \cdot \vec{i}_\alpha, \quad w^K \circ T_K = \tilde{w}, \quad \theta_\alpha^K \circ T_K = \tilde{\theta}_\lambda \vec{g}_\lambda \cdot \vec{i}_\alpha.$$

  is employed when enforcing the continuity of the displacements between elements.
- **The geometric curvatures can be calculated from the interpolated normal vector $\vec{n}_h$ as**

$$b_{\alpha\beta} \approx -\vec{i}_\alpha \cdot \vec{n}_{h,\beta}$$

Computer methods

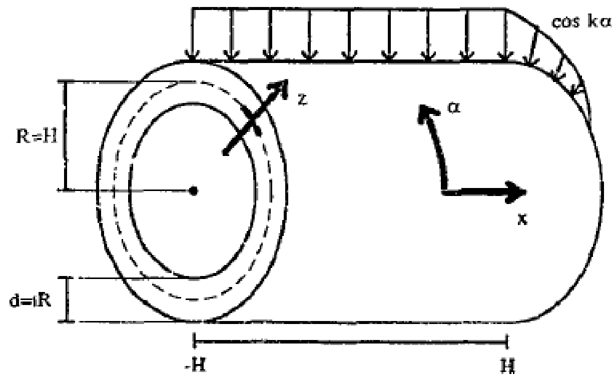
- Surface meshes generated by Gmsh, manually, or imported from other software
- Current implementation in Wolfram language (Mathematica)
- Can handle about million degrees of freedom

Numerical Results

Pitkäranta's cylinder

J. Pitkäranta et. al. CMAME 128 (1995), pp. 81-121

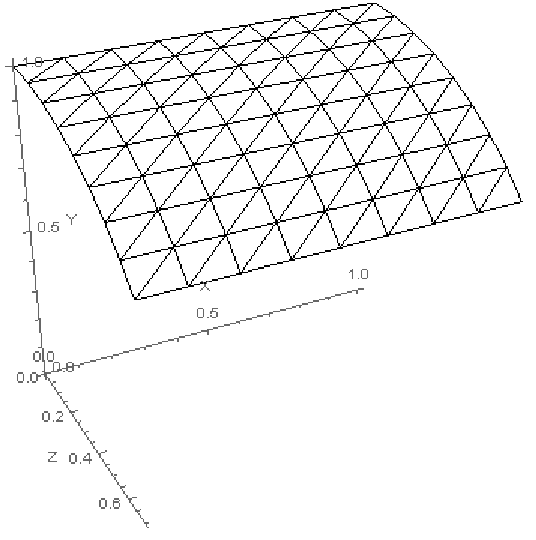
- **Cylindrical shell with half-length *H* = radius *R:***



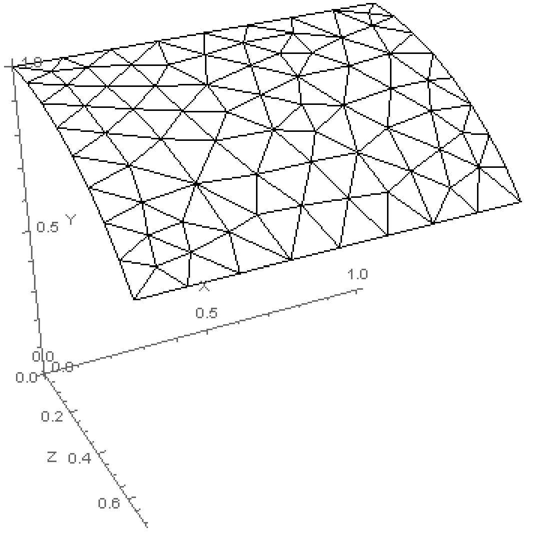
- ***Features all asymptotic categories of shell deformations*:**
  - Clamped ends -> Membrane-dominated
  - Free ends -> Bending-dominated
  - Simply supported ends -> Intermediate state (edge effect dominates)

Mesh sequences

We consider two mesh sequences with *N* elements per edge:



Uniform



Delaunay

Error indicator

- **We measure the quality of the numerical solution using the *strain energy error indicator***

$$e(h) = \sqrt{\frac{|U_{\mathrm{ref}} - U_h|}{U_{\mathrm{ref}}}}, \quad h = \frac{1}{N}$$

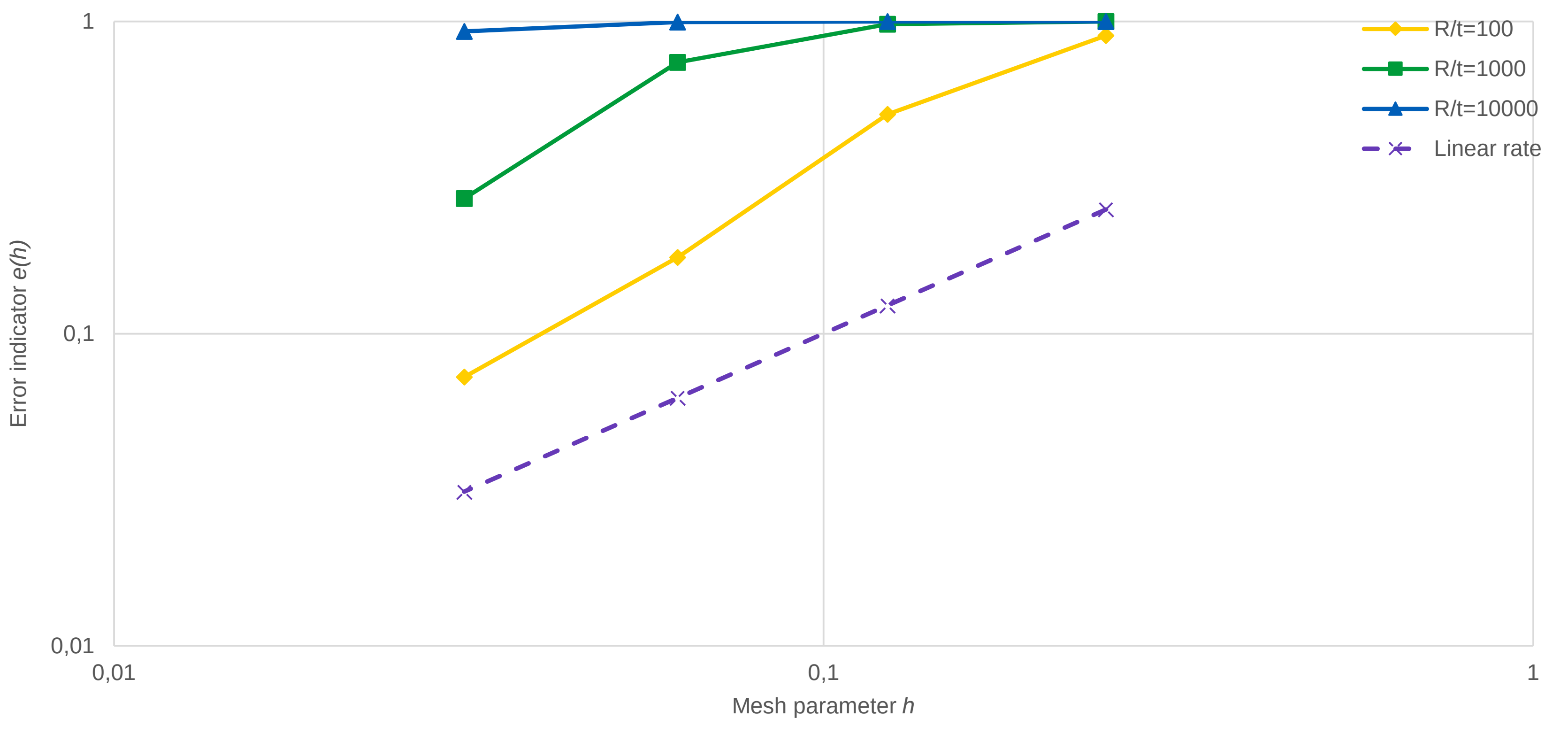
- **This error indicator is not completely reliable for non-conforming FE methods, but its computation is very easy**

- **Ideally, for our linear elements, we should have**

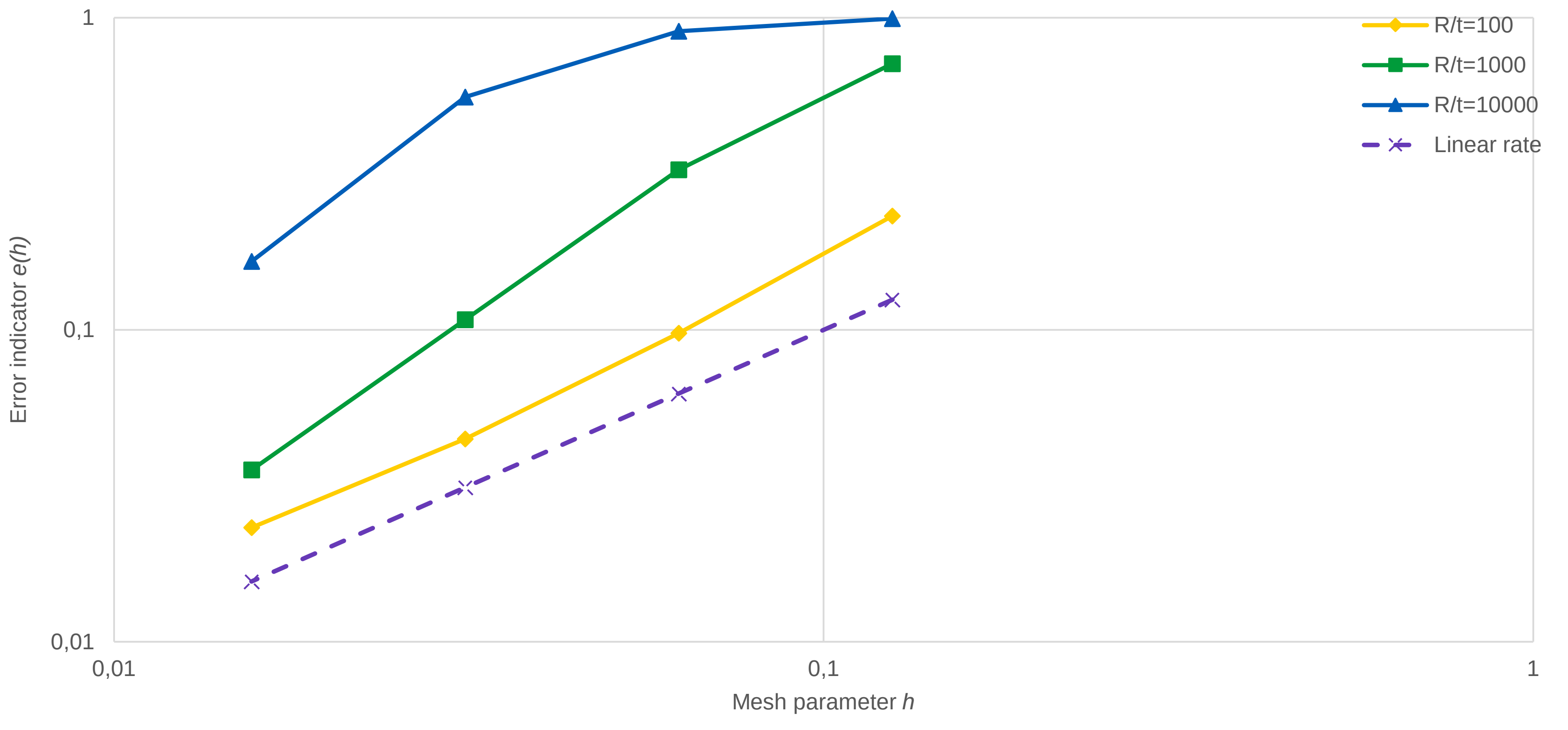
$$e(h) \leq Ch,$$

  where the constant C **does not depend on the ratio** R/t

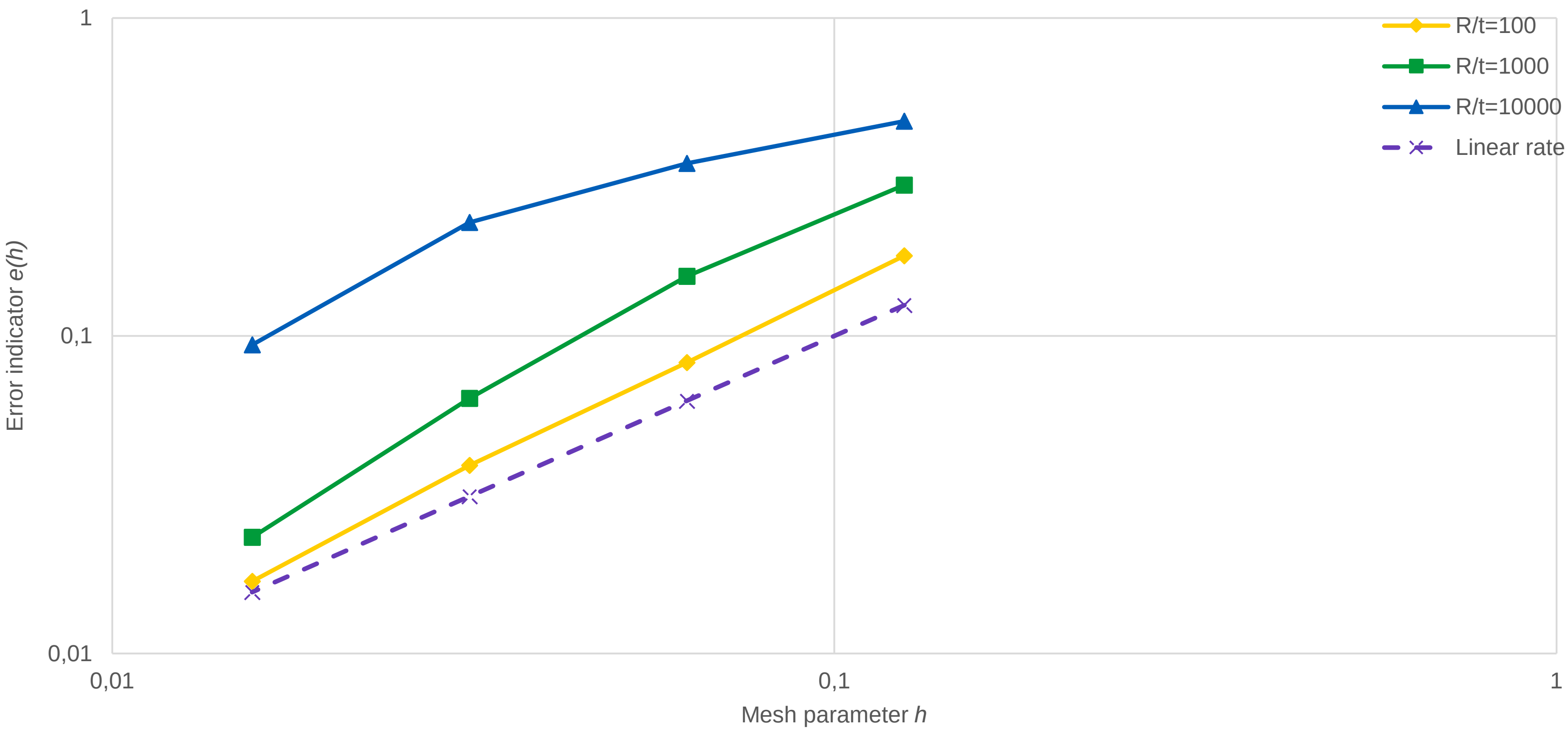
Cylinder with free ends (bending) COMSOL (linear triangular)



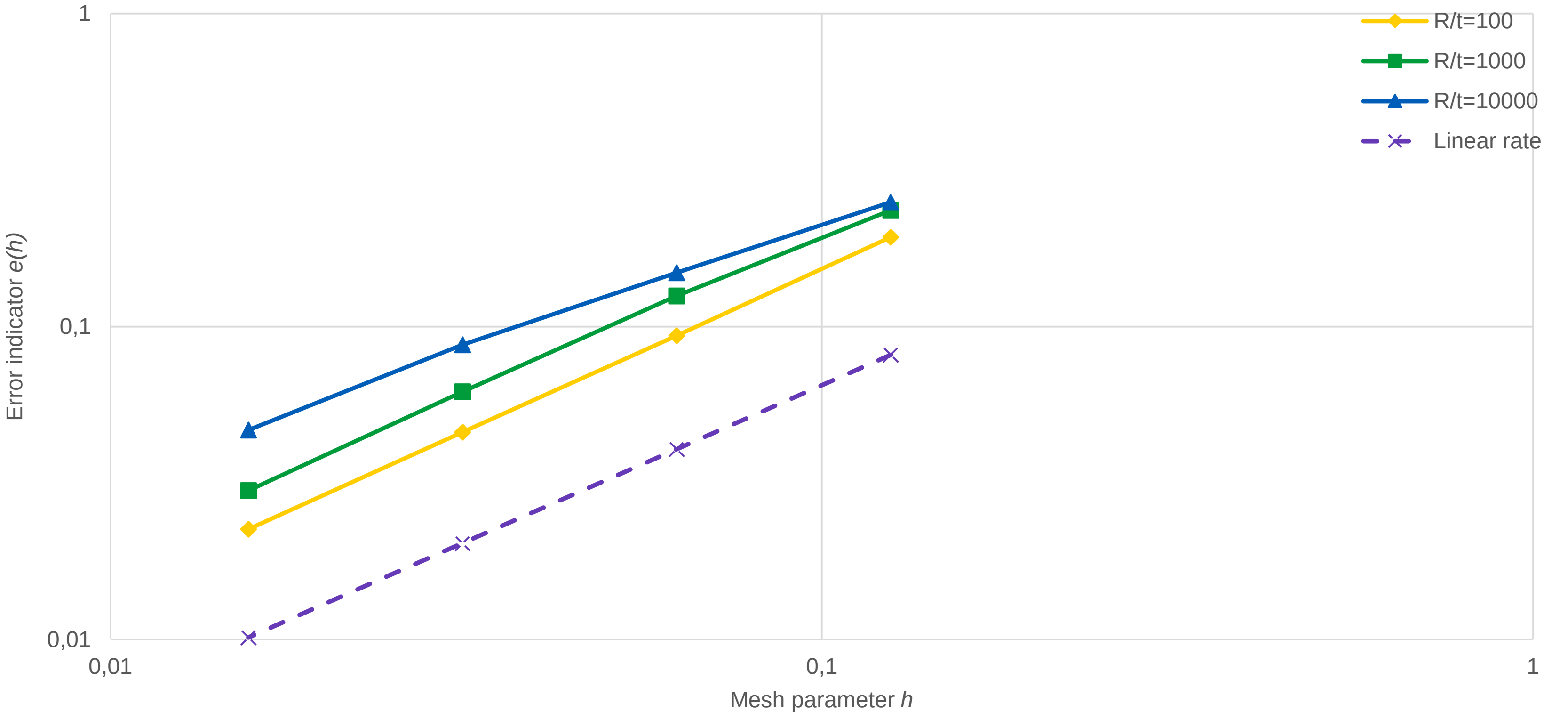
Cylinder with free ends (bending) MITC3S/Delaunay triangulation



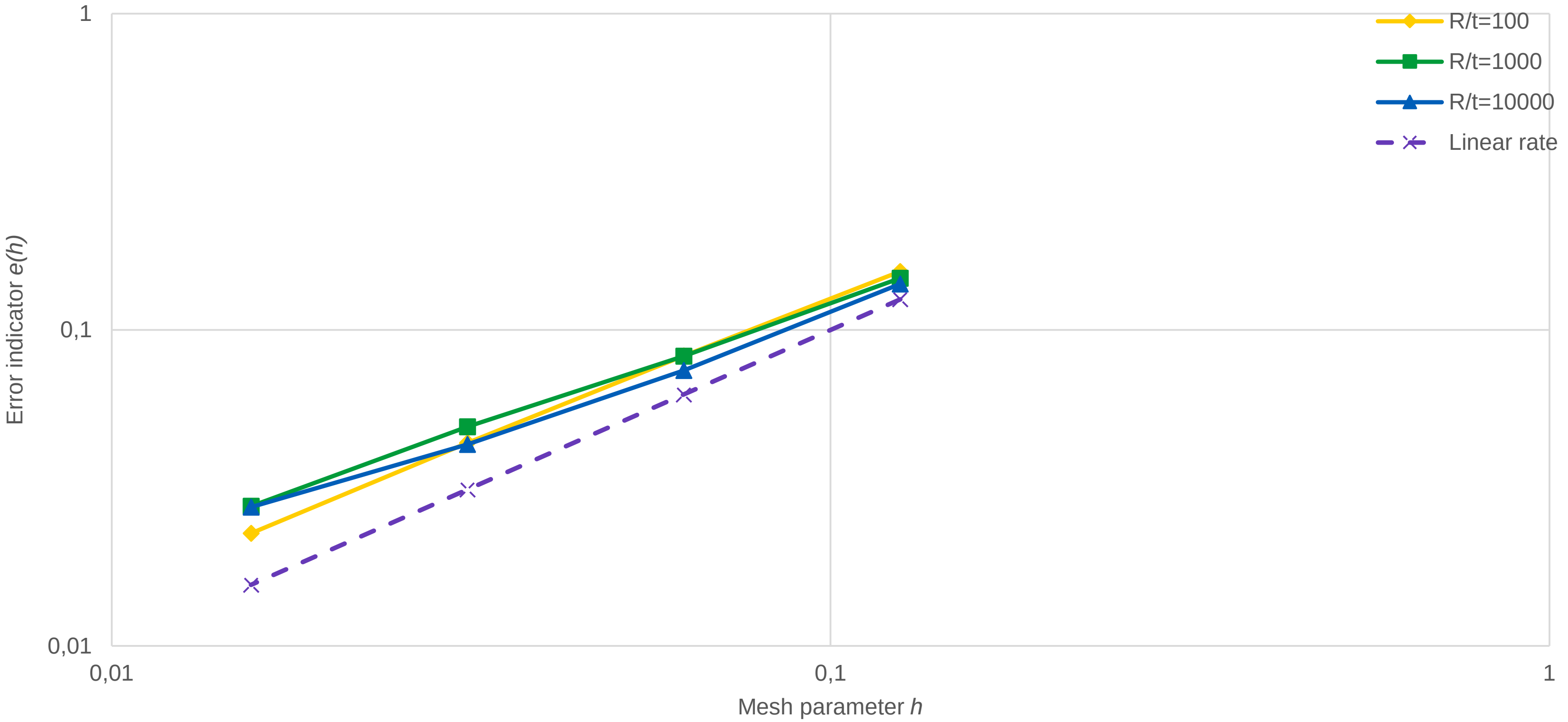
Cylinder with free ends (bending) Stabilized MITC3S/Delaunay



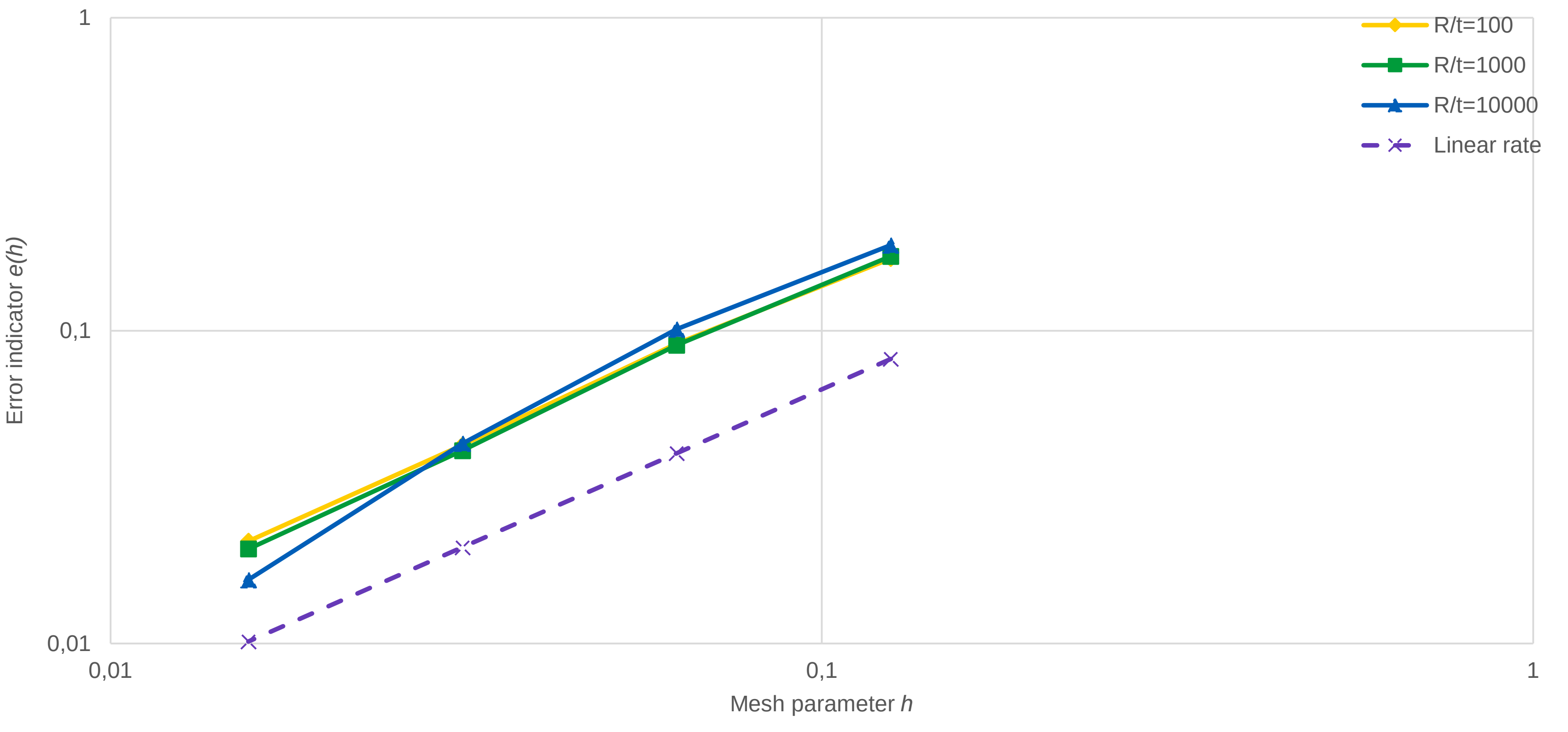
Cylinder with clamped ends MITC3S/Uniform



Cylinder with free ends (bending) MITC3S/Uniform



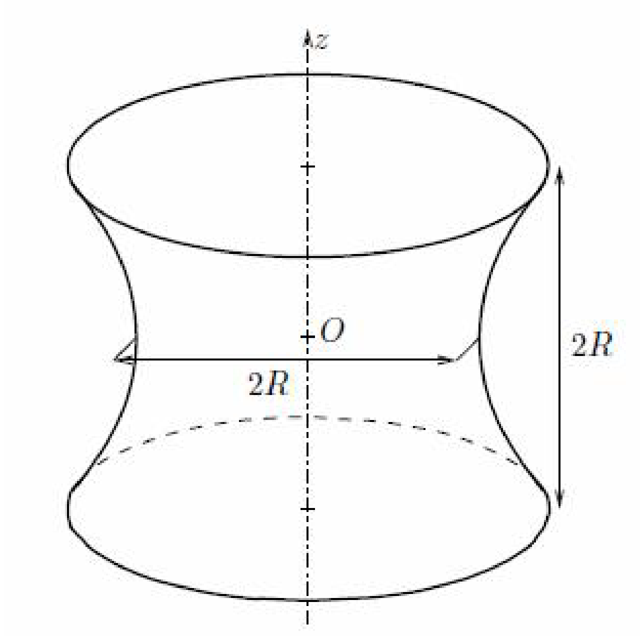
Cylinder with clamped ends Stabilized MITC3C/Delaunay

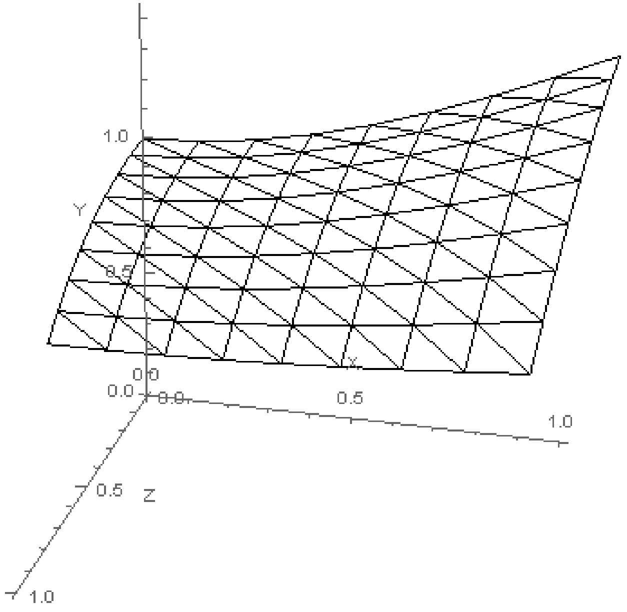


Doubly curved hyperboloid

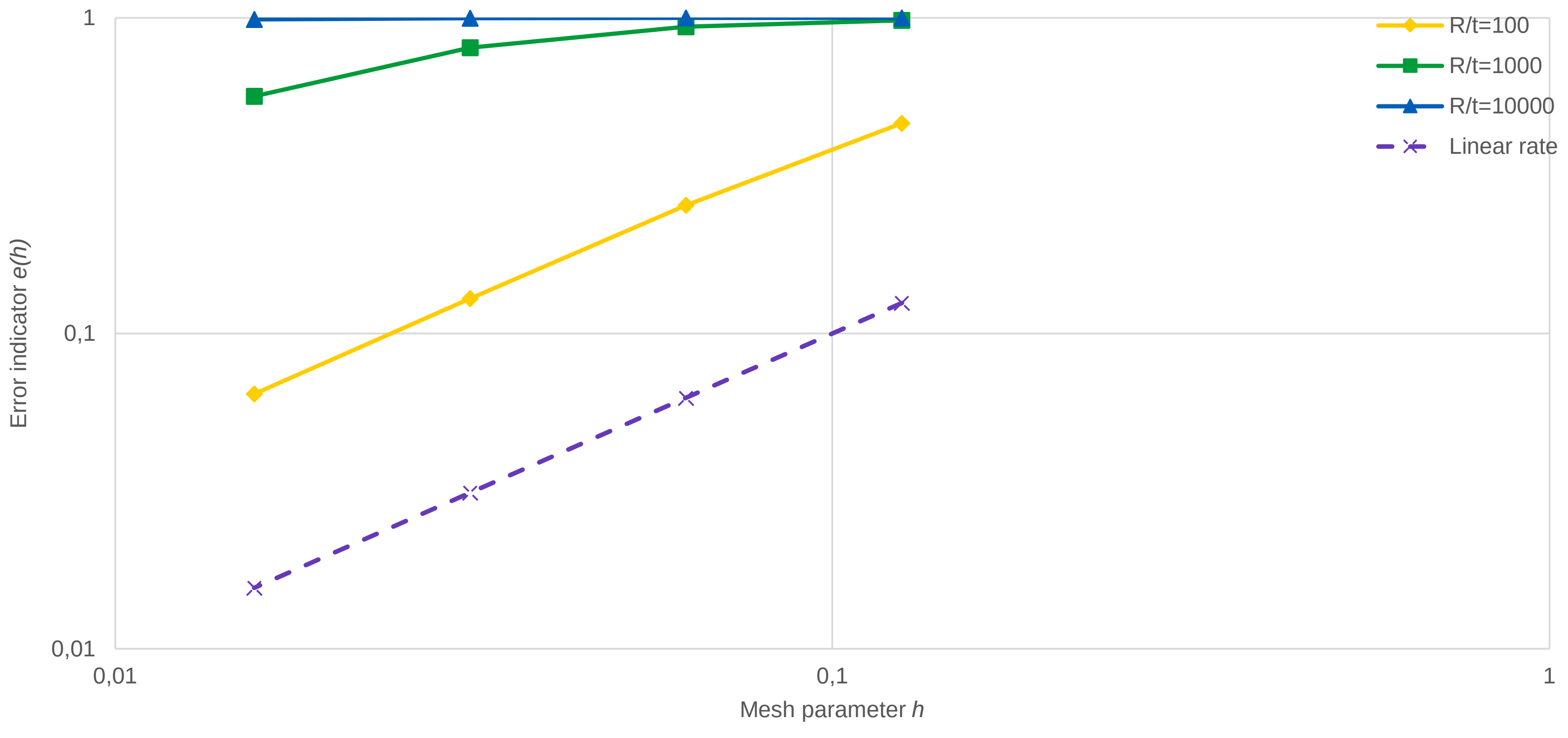
Hiller and Bathe, COMPUT STRUCT 81 (2003), pp. 639–654

- **Similar problem setup as in Pitkäranta's cylinder but the Gaussian curvature is not zero:**





Hyperboloid with free ends (bending) Stabilized MITC3S/Uniform mesh



Concluding Remarks

- **We have presented a family of triangular shell elements based on shell theory.**

- **The formalism allows explicit reduction of both membrane and transverse shear strains.**

- **The (stabilized) MITC3S element can provably approximate the inextensional modes of circular cylindrical shells when the mesh is aligned with the axis of the cylinder.**