Vortex Structures in Superfluids: Examples in $^3$He

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Simple superfluids

Order parameter is a scalar
- $^4$He
- many superconductors Al, Nb, Pb, ...
- many condensates of dilute gases

$$\Psi(\mathbf{r}) = \Psi_0 \exp[i\phi(\mathbf{r})] \quad v_s = \frac{\hbar}{M} \nabla \phi$$
\[ \Psi(r, \varphi, z) = \Psi_0 e^{i\varphi} C(r) \]

\[ \mathbf{v}_s = \frac{\hbar}{M} \nabla \varphi = \frac{\hbar}{M} \frac{\hat{\varphi}}{r} \]

\[ \kappa = \oint dl \cdot \mathbf{v}_s = \frac{\hbar}{M} \]

Charged superfluid: Flux line
In equilibrium the average superfluid velocity has to be equal to the normal fluid velocity

Line density $n$

\[ \int dl \cdot v_s = 2\pi R\Omega R = n\kappa\pi R^2 \]

\[ \Rightarrow n = \frac{2\Omega}{\kappa}. \]
Vortex lattice

**Flux lines in superconductors**
(NbSe$_2$, Ø. Fisher et al)

**Vortex lines in $^4$He**
(Yarmchuk et al 1979)

**Vortex lines in BEC**
(Abo-Shaeer et al 2001)

⇒ hexagonal lattice + boundary distortion

Are there other lattice structures?

- underlying crystal lattice
  → different vortex lattice structures, or no periodicity
- more complicated order parameter
Multi-component order parameter

\[ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \end{pmatrix} \]

- mixture BEC: two species of atoms
- spinor BEC: atoms identical but in different hyperfine states
- “exotic” superconductors
- \(^3\)He

Spin 1 condensate. Use basis \(|F = 1, m_F\rangle\)

\[ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix} = \sqrt{n}e^{i\phi} \begin{pmatrix} \zeta_1 \\ \zeta_0 \\ \zeta_{-1} \end{pmatrix} \]

\[ \langle S \rangle = \zeta_\alpha^* S_{\alpha\beta} \zeta_\beta \]

Two alternatives
1) Ferromagnetic state, \(|\langle S \rangle| = 1\). Characterized by direction of \(\langle S \rangle\)

2) Polar state, \(\langle S \rangle = 0\). Convenient representation in basis

\[ \Psi = \sqrt{n}e^{i\phi} \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} \]

\[ \frac{1}{\sqrt{2}} \left( -|1, 1\rangle + |1, -1\rangle \right) \]

\[ \frac{i}{\sqrt{2}} \left( |1, 1\rangle + |1, -1\rangle \right) \]

\[ |1, 0\rangle \]

\[ v_s = \frac{\hbar}{M} \nabla \phi \]

The variation of \(d\) and \(\langle S \rangle\) give rise to spin currents
## Symmetries of periodic vortex lattices in absence of crystal lattice effect

<table>
<thead>
<tr>
<th>Bravais lattice</th>
<th>space group</th>
<th>material</th>
<th>vortex name</th>
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<tbody>
<tr>
<td>hexagonal</td>
<td>$P6/mmm'$</td>
<td>$^4\text{He-II}$</td>
<td>vortex line</td>
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<tr>
<td></td>
<td></td>
<td>s.c. metals</td>
<td>flux line</td>
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<td>$^3\text{He-B}$</td>
<td>A-phase-core v.</td>
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<tr>
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<td>dilute gas</td>
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<tr>
<td>square</td>
<td>$P4/nbm'$</td>
<td>$^3\text{He-A}$</td>
<td>locked vortex 1 (LV1)</td>
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<tr>
<td>prim. rectang.</td>
<td>$Pb'\text{a}'n$</td>
<td>$^3\text{He-A}$</td>
<td>vortex sheet (VS)</td>
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<tr>
<td>cent. rectang.</td>
<td>$Cm'm'2$</td>
<td>$^3\text{He-B}$</td>
<td>double-core v.</td>
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<tr>
<td></td>
<td>$C2'$</td>
<td>$^3\text{He-A}$</td>
<td>cont. unlocked v. (CUV)</td>
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<td>$^3\text{He-A}$</td>
<td>locked vortex 2</td>
</tr>
<tr>
<td></td>
<td>$Cm'$</td>
<td>$^3\text{He-A}$</td>
<td>singular vortex (SV)</td>
</tr>
</tbody>
</table>
Basis for symmetry classification

coordinates \( r_i \), momenta \( p_i = m_i v_i \)

\[
H = \sum_i \frac{p_i^2}{2m_i} + V + \frac{1}{2} I \Omega^2 + U
\]

interactions confining potential and its kinetic energy (angular velocity \( \Omega \))

require \( L_{\text{tot}} = \sum_i r_i \times p_i + I \Omega = \text{constant} \)

\[
\Rightarrow H = \sum_i \frac{1}{2m_i} \left[ p_i - m_i v_n(r_i) \right]^2 + V - \frac{1}{2} \sum_i m_i v_n(r_i)^2 + U
\]

\( \Rightarrow \) periodic solutions

\( \Rightarrow \) neglect

charged superfluid \( p_i \rightarrow p_i - e_i A(r_i) \)

\[
H = \sum_i \frac{1}{2m_i} \left[ p_i - m_i v_n(r_i) - e_i A(r_i) \right]^2 + V - \frac{1}{2} \sum_i m_i v_n(r_i)^2 + U
\]

\( \Rightarrow \) London moment

(Equivalence of magnetic field and rotation ⇒ London moment)
Space groups for vortex lattices

There are 230 space groups for 3D crystals

Basic symmetry operations for vortex lattices:
1) translations
2) rotations around $\Omega$ (1,2,3,4,6)
3) rotation by $\pi$ around axis $\perp \Omega$ combined by time inversion (2’)
4) reflection in plane $\perp \Omega$ (m)
5) reflection in plane containing $\Omega$ combined by time inversion (m’)

vortex lattice: continuous translation symmetry along the rotation axis (?)
- space groups can still be used, but cubic groups are not possible

Why not 17 plane groups?
- they do not contain operation 4 above
- do not allow breaking of the continuous translation symmetry along rotation axis

Why not 1651 magnetic space groups?
- too complicated since time inversion appears trivially

(Karimäki & Thuneberg, PRB 1999)
An example: P4/nb’m’

Continuous locked vortex in $^3\text{He}$-A (Fujita, Nakahara, Ohmi & Tsuneto 1978)

The dashes show projection of $l$ vector on the plane (arrow heads are removed for clarity)

$l$ points up at the circular points and down at the hyperbolic points

equivalent points of the Bravais lattice
Second example: C2’

Continuous unlocked vortex in $^3\text{He-A}$ (Volovik & Seppälä 1983)

Centered rectangular lattice
Topological classification

In contrast to symmetry classification, topological classification depends on the type of the order parameter

Simple superfluids: circulation around a vortex

\[ \kappa = \oint dl \cdot v_s = \frac{\hbar}{M} \oint dl \cdot \nabla \varphi = n \frac{\hbar}{M}. \]

Many-component superfluids: additional topological invariants

\[ \nu_l = \frac{1}{4\pi} \int dx \, dy \, \mathbf{\hat{l}} \cdot \frac{\partial \mathbf{\hat{l}}}{\partial x} \times \frac{\partial \mathbf{\hat{l}}}{\partial y} \]

counts how many times \( \mathbf{l} \) sweeps the unit sphere
Order parameter of $^3$He

$^3$He is spin 1/2 fermion, fermions form pairs in state $S=1$, $L=1$. Such a Cooper pair state has macroscopic occupation, similar as in Bose condensation.

Spin wave functions ($S=1$)

- $S_x = 0$: $(-\uparrow\uparrow + \downarrow\downarrow)$
- $S_y = 0$: $i(\uparrow\uparrow + \downarrow\uparrow)$
- $S_z = 0$: $(\uparrow\downarrow + \downarrow\uparrow)$

Orbital wave functions ($L=1$)

$\Psi_{ij}(r)$ is the wave function for the center of mass of a Cooper pair.

In the A phase energetics limits the order parameter to the form

$\Psi_{\mu j} = \Delta \hat{d}_\mu (\hat{m}_j + i\hat{n}_j)$
Superfluid velocity in $^3\text{He-A}$

A phase factor $e^{i\phi}$ corresponds to rotation of $\hat{m}$ and $\hat{n}$ around $\hat{l}$:

$$e^{i\phi}(\hat{m} + i\hat{n}) = (\cos \phi + i \sin \phi)(\hat{m} + i\hat{n})$$
$$= (\hat{m} \cos \phi - \hat{n} \sin \phi) + i(\hat{m} \sin \phi + \hat{n} \cos \phi)$$

Superfluid velocity

$$\mathbf{v}_s = \frac{\hbar}{M} \nabla \phi = \frac{\hbar}{M} \sum_j \hat{m}_j \nabla \hat{n}_j.$$  

$\Rightarrow \mathbf{v}_s$ is coupled to the orientation of the triad $\hat{m}, \hat{n}, \hat{l}$  

$\Rightarrow$ Mermin-Ho relation $n = 2\nu_l$  

$\Rightarrow$ The structures presented above are vortices, although the order parameter vanishes nowhere
Vortex sheet

Vortex sheet is a tangential discontinuity in superfluid velocity

Vortex sheets were discussed before vortex lines:
Onsager 1948, Landau & Lifshitz 1955

Vortex sheets are unstable in simple superfluids

Vortex sheets can be stabilized in multi-component superfluids:
If there are two degenerate but distinct states, there is a domain wall between these.
This domain wall may trap vortices and thus becomes vortex sheet

Example in $^3$He-A

\[
\hat{d} \approx \text{constant}
\]

\[
f_D = -\frac{1}{2} \lambda_D (\hat{d} \cdot \hat{l})^2
\]

\hat{l} \text{ field:}

[Diagram showing the stabilization of vortex sheets in $^3$He-A with domain walls and solitons]
Structure of the sheet

Space group Pb’a’n
The equilibrium configuration is determined by the minimum of

\[ F \propto \int d^3 r \frac{1}{2} \rho_s (v_n - v_s)^2 + \sigma A \]

The equilibrium distance \( b \) between sheets

\[ b = \left( \frac{3\sigma}{\rho_s \Omega^2} \right)^{1/3} \]

The total area of the sheet

\[ A \propto \frac{1}{b} \propto \Omega^{2/3} \]

Connection lines with the side wall allow the vortex sheet to grow and shrink when angular velocity changes.
Playing with the sheet

Minimum energy state in a rectangular container

Stability of one concentric vortes sheet in a cylinder

Stability of one concentric vortes sheet in a cylinder

Local stability of a centered circular cylinder

Equilibrium
Superfluids with well defined phase

In the case the phase is uniquely defined, the vorticity has line structure, similar to simple superfluids.

The structure of the vortex core can be more complicated than in simple superfluids.

Example: $^3$He-B

The double-core structure (left)

The double-core can be interpreted as two half-quantum vortices.
Double core structure in an orifice

The double core structure in an orifice can explain the $\pi$ state observed in 3He-B Josephson junctions.

![Diagram of double core structure in an orifice with half-quantum vortex and current vs. phase diagram.](image)
Spin-current vortices

The vortices discussed above are characterized by mass (or electric) current.

In many-component systems, the variation of spin variables gives rise to spin current.

⇒ Spin-current vortex

Example: $^3$He-B

$$\Psi = \Delta \exp(i\phi) \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

where the rotation matrix $R(n, \theta)$ is parameterized by an angle $\theta$ and rotation axis $n$

Symmetry $U(1) \times SO_3$. Homotopy groups $\pi(U(1)) = Z$, $\pi(SO_3) = Z_2$.

Spin current is not conserved

Warning: no experimental evidence of this vortex yet
Combined spin and mass vortex, case 1

\[ q_A q / (\text{mass}) \text{ vortex} \]
\[ e_A e / \text{spin vortex} \]
\[ \Rightarrow F_{sm} < F_s + F_m \]

\( \theta = 104^\circ \)

\( \theta \neq 104^\circ \)

\( \approx 0.01 \mu m \)

Effect of non-conservation of spin current:
Combined spin and mass vortex, case 2

Assume an order parameter where phase is not uniquely defined

It is possible to have a line object where part of the change of the order parameter around the line comes from spin and part from the phase

Example: half quantum vortex in $^3$He-A

$$\Psi_{\mu j} = \Delta \hat{d}_\mu (\hat{m}_j + i\hat{n}_j)$$

The original order parameter is restored by combined operation:
1) turn $m$ and $n$ around $l$ by angle $\pi$ (meaning phase change by $\pi$)
2) turn $d$ by angle $\pi$

Half quantum vortex not favored in bulk $^3$He-A, but may be observed in a slab geometry (Mizusaki)

$\frac{1}{3}$ quantum vortex may appear in some spinor condensates (Machida)
Summary of vortex structures in $^{3}$He-A

Vortices in bulk liquid
- are these all at present experimental conditions?
- high rotation speeds (Kita)
- no proper calculation of the singular vortex

Vortices in restricted geometry
- various distorted forms of the bulk structures
- half quantum vortex?
- other structures?
Summary of vortex structures in $^3$He-B

Equilibrium vortices in bulk liquid: some observations suggest a third structure
- hysteresis at the vortex-core transition (Hall et al)
- gyroscopic experiments see similar but not same transition?

Spin-vortices observed as metastable states
Summary of vortex structures in dilute gases
Summary

Various vortex structures are possible for many-component order-parameter superfluids

These were illustrated by examples in superfluid $^3$He
Theory hierarchy of superfluid $^3$He

- **Quantum theory**
  - Spin = 1/2
  - Charge = 2 e
  - Mass = 5.01 $\times$ 10^{-27} kg
  - Gyromagnetic ratio = 2.04 $\times$ 10^8 1/Ts

- **Quasiclassical theory**
  - Density, Fermi velocity, interaction parameters, $T_c$, ...

- **Ginzburg-Landau theory**
  - $\alpha', \beta, K, \gamma$

- **Hydrodynamic theory**
  - Density, superfluid density, stiffness parameters, viscosity parameters
Vortex experiments in superfluid $^3$He

- Vortex sheet
- Singular vortex
- Soliton
- Locked vortex

NMR absorption

Frequency shift (kHz)

Larmor frequency

Potential energy for spin waves

Bound state

Vortex sheet

$\Psi$