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NOTE

Stability Analysis of a Keplerian Disk of Granular Grains: Influence of Thermal Diffusion

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We extend the investigation of the viscous stability of a dense planetary ring by the effect of thermal diffusion of random kinetic energy. We find that this additional diffusive process has mainly a stabilizing effect on the overstable modes. © 2000 Academic Press

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A poorly understood type of structure in Saturn's main rings, observed by the space probes, are the irregular structures in ring regions of high optical depth $\tau > 1$ at scales seen down to the limit of the resolution of the experiments (≈100 m). A favored physical explanation for these fluctuations has been the viscous instability first investigated by Lin and Bodenheimer (1981), Ward (1981), and Lukkari (1981). Viscous instability takes place when the derivative of dynamic shear viscosity with respect to density is negative which promotes the onset of a self-amplifying negative diffusion process. Araki and Tremaine (1986) have shown that the finite size of the ring particles and the related contributions to the transport coefficients suppress the viscous instability. This conclusion has been confirmed by numerical N-particle experiments (Wisdom and Tremaine 1988, Salo 1991, 1992a).

Another interesting effect, which is able to promote the formation of radial structures, is the viscous overstability (or pulsational instability) predicted in accretion disks (Kato 1978, Blumenthal *et al.* 1984) and in the context of planetary rings (Papaloizou and Lin 1988, Schmit and Tscharnuter 1995). Here, the system becomes unstable when the mentioned derivative of the dynamic shear viscosity with respect to density becomes large and positive (see, for instance, Papaloizou and Lin 1988, Schmit and Tscharnuter 1995). In the latter investigations a quasi-equilibrium between viscous heating and collisional cooling has been assumed, and only momentum and mass balances were used in the hydrodynamic modeling of the ring. Thus, the influence of (kinetic) temperature fluctuations was not taken into account. In the case of accretion disks an energy balance equation has been considered to model the influence of thermal radiation on the stability behavior of the disk (Kato 1978, Blumenthal *et al.* 1984). In this study we demonstrate the effect of kinetic heat transport inherent in the random motion of ring particles on overstability in a dense planetary ring. For simplicity we restrict our attention to purely axisymmetric perturbations of basic Keplerian flow of a ring. It must be noted, however, that nonaxisymmetric structures might be important for planetary rings, because in theoretical models and in simulations they occur naturally in the presence of the self-gravity of the disk on a 100-m scale (Julian and Toomre 1966, Salo 1992b; see also Goldreich and Tremaine 1978a). These structures may influence the formation of a mainly radial pattern as seen on larger scales by space probes, but even then the effect of kinetic heat flux can be expected to be preserved, although possibly quantitatively altered, and it should not be neglected in future hydrodynamic modeling of the ring flow.

We perform a linear stability analysis of the hydrodynamic equations and vary parameters, which serve as scalings of the constitutive relations for the transport coefficients. We use cylindrical coordinates (r, Θ, z) and assume an axisymmetric $(\partial/\partial\Theta \to 0)$, thin disk of granular grains surrounding a planet. The surface mass density of the disk is given by $\sigma = \int \rho\, dz$ (mass density $\rho(\mathbf{r}, t)$), the vertically integrated pressure is $P = \int p\, dz$, and the velocity field is denoted by $\mathbf{v} = u(t, r)\mathbf{e}_r + v(t, r)\mathbf{e}_\Theta$. With these assumptions the balances of momentum, mass, and energy read

$$\dot{u} + uu' - \frac{v^2}{r} = -\frac{GM}{r^2} - \Phi' - \frac{1}{\sigma}P' + \frac{2}{\sigma}(\eta u')' + 2\nu\left(\frac{u}{r}\right)' + \frac{1}{\sigma}\left[\sigma\left(\zeta - \frac{2}{3}\nu\right)\frac{(ru)'}{r}\right]' \tag{1a}$$

$$\dot{v} + \frac{u}{r}(rv)' = \frac{1}{\sigma r^2}\left[\eta r^3\left(\frac{v}{r}\right)'\right]' \tag{1b}$$

$$\dot{\sigma} = -\frac{(r\sigma u)'}{r} \tag{1c}$$

$$\frac{3}{2}\sigma(\dot{T} + uT') = \frac{1}{r}[r\kappa T']' - P\frac{(ru)'}{r} + \eta\left[2(u')^2 + \left[r\left(\frac{v}{r}\right)'\right]^2 + 2\left(\frac{u}{r}\right)^2\right] + \sigma\left(\zeta - \frac{2}{3}\nu\right)\left[\frac{(ru)'}{r}\right]^2 - \gamma, \tag{1d}$$

where the dots and the primes denote the partial derivatives with respect to the time t and the radial distance r from the central body. G, M, Φ, $\Omega = v/r$ are the gravitational constant, the mass of the central body, the gravitational potential of the disk, and the angular velocity, respectively. T is the temperature defined via the trace of the velocity dispersion tensor of the ring particles. The vertically integrated pressure is labeled by $P = \sigma T$. The (vertically integrated) quantities ν, $\eta = \sigma\nu$, ζ, and κ are the kinematic shear viscosity, the dynamic viscosity, the kinematic bulk viscosity, and the heat conductivity, respectively.

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The difference between granular gases and the usual hydrodynamic gases or fluids is the granular cooling γ due to inelastic collisions as well as differences in the transport coefficients. The gravitational potential Φ is determined by the Poisson equation

$$\frac{1}{r}(r\Phi')' + \partial_z^2\Phi = 4\pi G\sigma\delta(z), \tag{2}$$

where $\delta(z)$ is Dirac's delta function. The right-hand sides of (1a) and (1b) contain the vertically integrated volume forces: the gravity of the central body, the self-gravity of the disk, pressure, and friction. The energy balance (1d) is governed by the effects of the heat conduction, mechanical work, the viscous heating, and the collisional cooling. To investigate the stability of the system we consider the vector $\mathbf{x} = (u, v, \sigma, T, \Phi)$ to be composed of a ground state $\mathbf{x}_0$ and small perturbations $\mathbf{x}_1$ according to

$$\mathbf{x} = \mathbf{x}_0(r_0) + \mathbf{x}_1(r, t), \tag{3}$$

with $|r_0 - r| \ll r_0$. All quantities $y(\mathbf{x})$ depending on the state vector $\mathbf{x}$, like the pressure P and all transport coefficients, are expanded about the ground state $\mathbf{x}_0$ according to $y = y_0 + \nabla_{\mathbf{x}} y|_0 \cdot \mathbf{x}_1 + O(|\mathbf{x}_1|^2) \approx y_0 + y_1$. The stationary ground state is characterized by the approximate solution

$$v_0 = \left(\frac{GM}{r_0}\right)^{\frac{1}{2}} \equiv \Omega_0 r_0; \quad T_0' = \sigma_0' = u_0 = 0; \quad \partial_t \mathbf{x}_0 = 0. \tag{4}$$

Substituting Eqs. (3) and (4) into (1a)–(1d), we obtain the linearized system

$$\dot{u}_1 - 2\Omega_0 v_1 = -\Phi_1' - \frac{P_1'(\sigma_1, T_1)}{\sigma_0} + \left(\zeta_0 + \frac{4}{3}\nu_0\right)u_1'' \tag{5a}$$

$$\dot{v}_1 + \frac{1}{2}\Omega_0 u_1 = -\frac{3}{2}\frac{\Omega_0}{\sigma_0}\eta_1'(\sigma_1, T_1) + \nu_0 v_1'' \tag{5b}$$

$$\dot{\sigma}_1 + \sigma_0 u_1' = 0 \tag{5c}$$

$$\frac{3}{2}\sigma_0\dot{T}_1 = \frac{1}{r}(r\kappa_0 T_1')' - P_0\frac{(ru_1)'}{r} + 3\Omega_0\left\{\frac{3}{4}\eta_1\Omega_0 - \eta_0 r\left(\frac{v_1}{r}\right)'\right\} - \gamma_1(\sigma_1, T_1) \tag{5d}$$

$$\Phi_1'' + \partial_z^2\Phi_1 = 4\pi G\sigma_1\delta(z), \tag{6}$$

where $O(|\mathbf{x}_1|^2)$ are neglected. We perform a Fourier decomposition into purely radial modes $x_1 \propto \exp(st + ikr)$ for all perturbed quantities, where s and $k = 2\pi/\lambda$ are the growth rate and wave number (λ radial wavelength), respectively. Then, with Eqs. (5a)–(5d) the solubility condition of the resulting algebraic system yields the dispersion relation

$$s^4 + As^3 + Bs^2 + Cs + D = 0, \tag{7}$$

with

$$A = \frac{1}{\sigma_0}\left(\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0 - \frac{3}{2}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right) + k^2\frac{1}{\sigma_0}\left(\frac{7}{3}\eta_0 + \frac{2}{3}\kappa_0 + \sigma_0\zeta_0\right)$$

$$\begin{aligned} B = {} & \Omega_0^2 - k2\pi G\sigma_0 + k^2\left(\frac{\partial P}{\partial\sigma}\bigg|_0 + \eta_0\frac{7}{3\sigma_0^2}\left\{\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0 - \frac{3}{14}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right\}\right. \\ & \left. + \frac{2}{3\sigma_0^2}\frac{\partial P}{\partial T}\bigg|_0 P_0 + \frac{\zeta_0}{\sigma_0}\left\{\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0 - \frac{3}{2}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right\}\right) \\ & + k^4\frac{1}{\sigma_0^2}\left(\eta_0\left\{\frac{4}{3}\eta_0 + \sigma_0\zeta_0\right\} + \frac{2}{3}\kappa_0\left\{\frac{7}{3}\eta_0 + \sigma_0\zeta_0\right\}\right) \end{aligned}$$

$$\begin{aligned} C = {} & \frac{1}{\sigma_0}\Omega_0^2\left(\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0 - \frac{3}{2}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right) - k2\pi G\left(\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0 - \frac{3}{2}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right) \\ & + k^2\left(\Omega_0^2\left\{3\frac{\partial\eta}{\partial\sigma}\bigg|_0 + \frac{2}{3\sigma_0}\kappa_0 - \frac{1}{\sigma_0^2}\frac{\partial P}{\partial T}\bigg|_0\eta_0 + \frac{2}{\sigma_0^2}\frac{\partial\eta}{\partial T}\bigg|_0 P_0\right\}\right. \\ & + \frac{1}{\sigma_0}\frac{\partial P}{\partial\sigma}\bigg|_0\left\{\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0 - \frac{3}{2}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right\} - \frac{1}{\sigma_0}\frac{\partial P}{\partial T}\bigg|_0\left\{\frac{2}{3}\frac{\partial\gamma}{\partial\sigma}\bigg|_0\right. \\ & \left.\left. - \frac{3}{2}\frac{\partial\eta}{\partial\sigma}\bigg|_0\Omega_0^2\right\}\right) - k^3 2\pi G\left(\eta_0 + \frac{2}{3}\kappa_0\right) + k^4\left(\frac{1}{\sigma_0^3}\eta_0\left\{\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0\right.\right. \\ & \left. + \frac{3}{2}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right\}\left\{\frac{4}{3}\eta_0 + \sigma_0\zeta_0\right\} + \frac{2}{3\sigma_0^3}\frac{\partial P}{\partial T}\bigg|_0\eta_0 P_0 + \frac{1}{\sigma_0}\frac{\partial P}{\partial\sigma}\bigg|_0\eta_0 \\ & \left. + \frac{2}{3\sigma_0}\frac{\partial P}{\partial\sigma}\bigg|_0\kappa_0\right) + k^6\frac{2}{3\sigma_0^3}\eta_0\kappa_0\left(\frac{4}{3}\eta_0 + \sigma_0\zeta_0\right) \end{aligned}$$

$$\begin{aligned} D = {} & k^2\frac{2}{\sigma_0}\Omega_0^2\left(\frac{\partial\eta}{\partial\sigma}\bigg|_0\frac{\partial\gamma}{\partial T}\bigg|_0 - \frac{\partial\eta}{\partial T}\bigg|_0\frac{\partial\gamma}{\partial\sigma}\bigg|_0\right) - k^3\frac{1}{\sigma_0}2\pi G\eta_0\left(\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0\right. \\ & \left. + \frac{3}{2}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right) + k^4\left(\frac{1}{\sigma_0^2}\frac{\partial P}{\partial\sigma}\bigg|_0\eta_0\left\{\frac{2}{3}\frac{\partial\gamma}{\partial T}\bigg|_0 + \frac{3}{2}\frac{\partial\eta}{\partial T}\bigg|_0\Omega_0^2\right\}\right. \\ & \left. - \frac{1}{\sigma_0^2}\frac{\partial P}{\partial T}\bigg|_0\eta_0\left\{\frac{2}{3}\frac{\partial\gamma}{\partial\sigma}\bigg|_0 + \frac{3}{2}\frac{\partial\eta}{\partial\sigma}\bigg|_0\Omega_0^2\right\} + \frac{2}{\sigma_0}\frac{\partial\eta}{\partial\sigma}\bigg|_0\Omega_0^2\kappa_0\right) \\ & - k^5\frac{4}{3\sigma_0}\eta_0 G\pi\kappa_0 + k^6\frac{2}{3\sigma_0^2}\frac{\partial P}{\partial\sigma}\bigg|_0\eta_0\kappa_0. \end{aligned}$$

Concerning the self-gravity we have used $\Phi_1(r, z = 0) = -2\pi G\sigma_1/k$, which follows from the jump condition of the potential at $z = 0$ as well as from the transition of the Poisson equation (2) at $z = 0$ to a Laplace equation for $z \neq 0$ since their solutions at $z \to \pm 0$ must coincide (see Shu 1984).

For an application of the dispersion relation (7) to the dense saturnian rings we will have to know the values for the dependency of the transport coefficients and the cooling on density and temperature. Since these are in general not yet available for dense rings we will limit our present study to the expected qualitative effects of the additional energy equation, by using a simplified but physically reasonable set of assumptions. For the constitutive relations we use $\nu \propto T^\alpha\sigma^\beta$, $\kappa \propto \eta$, and $\zeta \propto \nu$. According to analytical studies (Goldreich and Tremaine 1978b, Stewart *et al.* 1984) we use $\alpha = 1$. However, simulations of dense rings suggest an effective $\alpha \leq 1/2$ (see Salo 1991 and under the additional influence of a size distribution Salo 1992b). This effect can be attributed to the nonlocal part of the shear viscosity that depends much more weakly on the temperature than the local part. Thus, for the total viscosity the value of α may be reduced in dense systems. For the exponent β describing the dependence on density we have $\beta > 1$ (Araki and Tremaine 1986) in dense rings ($\tau > 1$). For the granular cooling we use $\gamma = \Omega\tau\sigma(1 - \epsilon^2)T$ (Stewart *et al.* 1984), where ϵ is the coefficient of restitution and $\tau = \pi R_p^2\sigma/m$ is the optical depth with the particle radius R_p and mass m. In principle, in an ensemble of inelastic particles the temperature may depend on the density as $T = c_0^2(\sigma/\sigma_0)^\delta$, where c_0 is the mean velocity dispersion. In force-free granular gases a value of $\delta = -2$ is observed (Haff 1983, Jenkins and Richman 1985). For $\delta < -1$ another instability may arise, the so-called pressure instability of granular gases (Goldhirsch and Zanetti 1993) where $\partial P/\partial\sigma < 0$ (Spahn *et al*. 1997, Petzschmann *et al*. 1999). This instability occurs only at very small scales. For Keplerian systems simulations indicate a small negative δ for dilute systems, vanishing for increasing τ (Salo 1991, see Fig. 5a). This is in accordance with the simulational result $\partial\log(\nu)/\partial\log(\sigma) \equiv \beta + \alpha\delta = 1.27$ (Wisdom and Tremaine 1988, Salo 1991) when compared to the aforementioned analytically determined $\beta > 1$ (Araki and Tremaine 1986) for $\tau > 1$, suggesting also a small value of $|\delta|$. Therefore, we choose $\delta = 0$ in our plots.

For the ratio of heat conductivity to shear viscosity we take the value for dense (elastic) hard sphere gases (Chapman and Cowling 1970), i.e., $\kappa/\eta = 5$, which should give at least the right order of magnitude for the dissipative ring particle ensemble. Similarly, we find for the ratio of bulk viscosity to shear viscosity $\zeta/\nu = 1$.

The limit $\kappa \to \infty$ yields the dispersion relation derived by Schmit and Tscharnuter (1995) provided that the pressure is given by $P = \sigma T_0$ and $T_0 = c_0^2$. This becomes clear by splitting off the hydrodynamic heat flux mode. Then, the dispersion relation can be rearranged as

$$
\begin{aligned}
0 = \left(s + \frac{2}{3\sigma_0}k^2\kappa_0\right) \times \Bigg[s^3 + s^2 \frac{k^2}{\sigma_0}\left(\frac{7}{3}\eta_0 + \sigma_0\zeta_0\right) \\
+ s\left(\Omega_0^2 - k2\pi G\sigma_0 + k^2 \frac{\partial p}{\partial \sigma}\bigg|_0 + k^4 \frac{1}{\sigma_0^2}\eta_0\left(\frac{4}{3}\eta_0 + \sigma_0\zeta_0\right)\right) \\
+ k^2\left(3\Omega_0^2 \frac{\partial \eta}{\partial \sigma}\bigg|_0 - k2\pi G\eta_0 + k^2\frac{1}{\sigma_0}\frac{\partial p}{\partial \sigma}\bigg|_0 \eta_0\right)\Bigg] + F\left(\frac{\partial \eta}{\partial T}\bigg|_0, \frac{\partial P}{\partial T}\bigg|_0, \frac{\partial \gamma}{\partial T}\bigg|_0\right),
\end{aligned}
\tag{8}
$$

where the function F contains all the derivatives with respect to temperature. For this isothermal limit the regions of stability and overstability are shown in Fig. 1 (dashed lines), in dependence of β and λ. For the parameters we have used

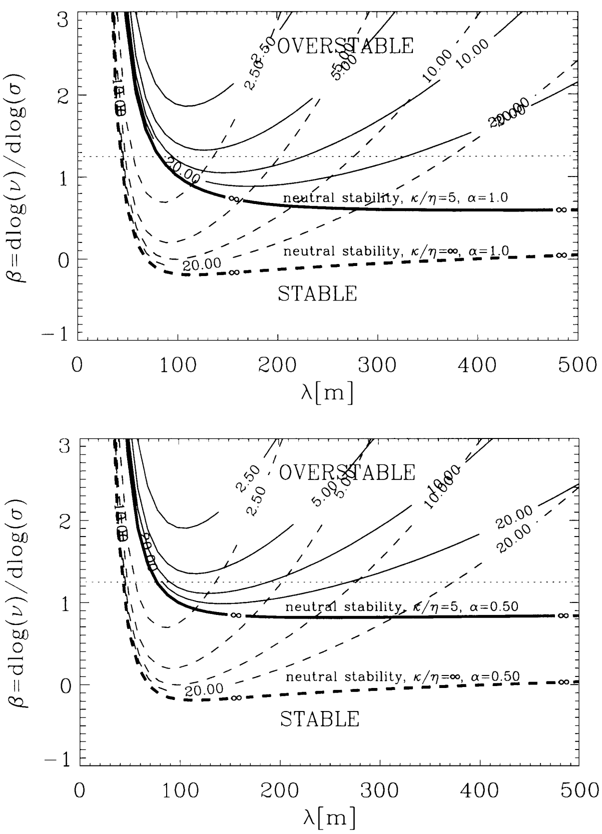

FIG. 1. The *overstable mode* of the dispersion relation (Eq. 7). Lines of equal 10-folding times of harmonic perturbations are shown in units of the orbital time in the (λ, β) plane. The isothermal case $\kappa/\eta = \infty$ (dashed curves) and the nonisothermal case $\kappa/\eta = 5$ (solid curves) are plotted. Neutral stability is drawn in each case as a thick line, separating the overstable (above the line) from the stable region (below the line). The upper plot corresponds to $\alpha\,(\equiv \partial \log(\nu)/\partial \log(T)) = 1$, the lower to a smaller α (see text). Here, the temperature is assumed to be independent of the surface density, i.e., $\delta = 0$. The optical depth is equal to 1. The horizontal dotted line marks the value of $\beta = 1.27$, corresponding to this optical depth in simulations of Wisdom and Tremaine (1988).

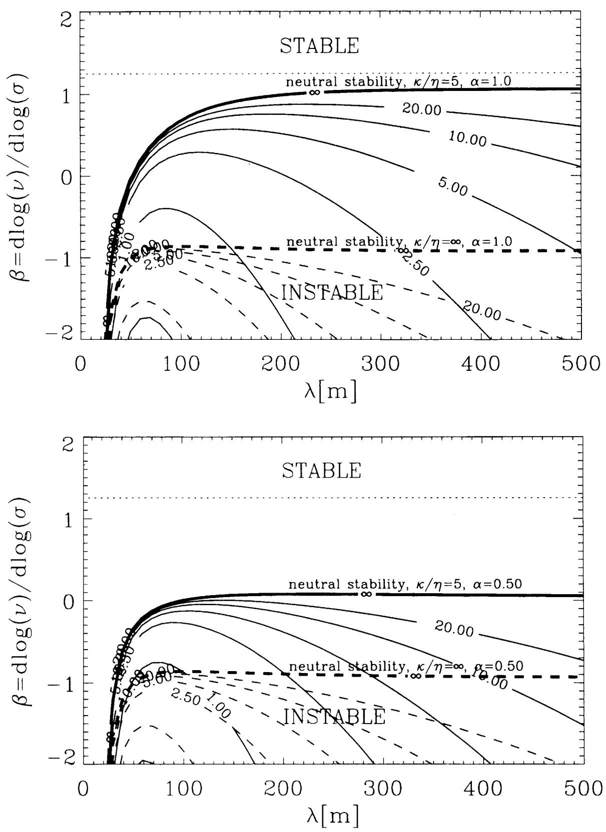

FIG. 2. The *instable mode* of the dispersion relation (Eq. 7) for the same parameters as described in the legend of Fig. 1.

$\Omega_0 = 1.95 \times 10^{-4}\ \mathrm{s}^{-1}$, $\sigma_0 = 133\ \mathrm{g\,cm}^{-2}$ corresponding to $\tau = 1$ for ice particles, $c_0 = 0.2\ \mathrm{cm\,s}^{-1}$, $\nu_0 = 54\ \mathrm{cm}^2\,\mathrm{s}^{-1}$, corresponding to the B ring of Saturn, and further a particle radius $R_p = 1$ m, a particle bulk density of $1\ \mathrm{g\,cm}^{-3}$, and $\epsilon = 0.4$ for the coefficient of restitution. The latter is the mean value resulting from the variable restitution law (Bridges *et al.* 1984) at the mentioned velocity dispersion. For $\beta = 1.27$ (corresponding to $\tau = 1$, $\delta = 0$) all wavelengths $\lambda > 40$ m should be overstable. The fastest growing wavelength is slightly less than 100 m.

With decreasing ratio κ/η we find that the neutral stability curve is shifted to higher β and to larger wavelengths (solid lines in Fig. 1). This behavior is not sensitive to small variations of κ/η. The lower and upper parts of Fig. 1 demonstrate an additional stabilizing effect of a reduced value of α for dense systems, as mentioned before, in the nonisothermal model. Another factor that tends to stabilize the overstable mode is the bulk viscosity. If ζ/ν exceeds unity (unity was assumed in Fig. 1), the viscous overstability is more difficult to achieve. For example, $\zeta/\nu = 2$ would shift the neutral boundary to about $\beta = 1$ for the case of $\kappa/\eta = 5, \alpha = 0.5$. The variation of the bulk viscosity affects similarly the isothermal model as already indicated by the analysis of Schmit and Tscharnuter (1995), although they assumed $\zeta/\nu = 1$ throughout their paper.

Figure 2 shows the influence of the heat flow on the instable mode ($\mathrm{Re}(s) > 0$, $\mathrm{Im}(s) = 0$) of Eq. (7). Here, for $\kappa/\eta = 5$ the instability is shifted to values of β higher than those for the isothermal case $\kappa/\eta = \infty$. However, this shift is less pronounced for smaller α. Thus, in dense systems ($\tau > 1$) with $\beta > 1.27$ viscous instability is also not to be expected in the nonisothermal model.

The result of this Note is that heat conduction is able to change the stability behavior of hydrodynamic models of dense planetary rings considerably. Further, the importance of a detailed knowledge of the magnitudes of the transport coefficients and their dependence on density *and* temperature is stressed. For instance, the differences observed in the viscosities for free granular gases (Jenkins and Richman 1985) and for the material forming a planetary ring (Goldreich and

Tremaine 1978b, Stewart *et al.* 1984, Salo 1991, Schmidt *et al.* 1999) show that the transport coefficients and $T(\sigma)$ are also determined by the physical environment. Further, an increased ratio ζ/ν additionally stabilizes our hydrodynamic model, shifting neutral stability to even higher values of β for the overstable mode. This emphasizes the importance of an improved analytical and numerical evaluation of the constitutive relations in order to study the stability of planetary rings in greater detail.

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