Example 10: Obtain the inverse Z transform of $X(z)=\frac{z(z+2)}{(z-1)^2}$, using partial fraction expansion method.

Solution:

If the $X(z)$ has a zero at the origin, it may be simpler to expand $\frac{X(z)}{z}$ into fractions in the current problem it gives:

$$\frac{X(z)}{z}=\frac{z+2}{(z-1)^2}=\frac{k_{11}}{(z-1)}+\frac{k_{12}}{(z-1)^2}$$

where

$$k_{11}(z-1)+k_{12}=z+2$$
$$\left.\begin{array}{r} k_{11}=1 \\ -k_{11}+k_{12}=2 \end{array}\right\}\Leftrightarrow k_{11}=1, k_{12}=3$$

resulting in

$$\frac{X(z)}{z}=\frac{1}{(z-1)}+\frac{3}{(z-1)^2} \qquad \text{and so} \qquad \underline{X(z)=\frac{z}{(z-1)}+\frac{3z}{(z-1)^2}}$$

Using the Table 1,

$$Z^{-1}\{X(z)\}=Z^{-1}\left\{\frac{z}{(z-1)}+\frac{3z}{(z-1)^2}\right\}=Z^{-1}\left\{\frac{z}{(z-1)}\right\}+Z^{-1}\left\{\frac{3z}{(z-1)^2}\right\}$$

$$x(kh)=1+3kh$$

and

$$x(t)=1+3t\,.$$

Applying the Z or the inverse Z transform, one must always remember the sampling time, h, which must be taken into account when calculating the values of a sampled data signal. Two different sampling times, h = 1sec and h = 0.5 sec were chosen. The values of the signal at the sampling instants are listed in the Table below.

k	**t (h=1 sec)**	**x(k*1)**	**t (h=0.5sec)**	**x(k*0.5)**
0	0 sec	1	0.0 sec	1.0
1	1 sec	4	0.5 sec	2.5
2	2 sec	7	1.0 sec	4.0
3	3 sec	10	1.5 sec	5.5
4	4 sec	13	2.0 sec	7.0
5	5 sec	16	2.5 sec	8.5
6	6 sec	19	3.0 sec	10

Example 11: Obtain the inverse Z transform of the following signal

$$X(z)=\frac{z}{z-1}\frac{z}{z-0.5}\,.$$

Solution:

The inverse Z transform is calculated as

$$x(k)=Z^{-1}\left\{\frac{z}{z-1}\frac{z}{z-0.5}\right\}$$

The partial fraction expansion method gives

$$\frac{X(z)}{z}=\frac{z}{(z-1)(z-0.5)}=\frac{k_1}{z-1}+\frac{k_2}{z-0.5}$$

$$\text{where}\quad k_1=\left.\frac{(z-1)z}{(z-1)(z-0.5)}\right|_{z=1}=2$$

$$k_1=\left.\frac{(z-0.5)z}{(z-1)(z-0.5)}\right|_{z=0.5}=-1$$

$$X(z)=\frac{k_1 z}{z-1}+\frac{k_2 z}{z-0.5}=\frac{2z}{z-1}-\frac{z}{z-0.5}\,.$$

The inverse Z transform results in

$$x(k)=Z^{-1}\left\{\frac{2z}{z-1}-\frac{z}{z-0.5}\right\}=Z^{-1}\left\{\frac{2z}{z-1}\right\}-Z^{-1}\left\{\frac{z}{z-0.5}\right\}.$$

Using Table 1,

$$2Z^{-1}\left\{\frac{z}{z-1}\right\}=2 \qquad \text{and}$$

$$Z^{-1}\left\{\frac{z}{z-0.5}\right\}=Z^{-1}\left\{\frac{z}{z-e^{-ah}}\right\}=e^{-akh}\text{, where}\quad a=-\frac{\ln 0.5}{h}\,.$$

The sampled data signal is thus

$$x(k)=2-e^{-\frac{\ln 0.5}{h}kh}=2-e^{-\ln 0.5 k}$$

The form of the original continuous-time signal shows clearly that the sampling must be taken into account to provide a unique solution:

$$x(t)=2-e^{-\frac{\ln 0.5}{h}t}\,.$$