Example 13: Pulse transfer function from differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{du}{dt} + 3u$$

a) without and b) with ZOH

derive the transfer function using Laplace transform

$$L\left\{\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y\right\} = L\left\{\frac{du}{dt} + 3u\right\}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = sU(s) + 3U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2 + 3s + 2}$$

partial fraction expansion

$$G(s) = \frac{s+3}{(s+1)(s+2)}$$

$$G(s)=\frac{s+3}{(s+1)(s+2)}$$

$$G(s)=\frac{k_1}{s+1}+\frac{k_2}{s+2}$$

$$k_1=\left.\frac{(s+1)(s+3)}{(s+1)(s+2)}\right|_{s=-1}=2$$

$$G(s)=\frac{2}{s+1}-\frac{1}{s+2}$$

$$k_2=\left.\frac{(s+2)(s+3)}{(s+1)(s+2)}\right|_{s=-2}=-1$$

without ZOH

$$Z\left\{\frac{2}{s+1}-\frac{1}{s+2}\right\}=Z\left\{\frac{2}{s+1}\right\}-Z\left\{\frac{1}{s+2}\right\}$$

Table 1 Z transform of signals

e^{-at}	$\frac{1}{s+a}$	e^{-anh}	$\frac{z}{z-e^{-ah}}$

using Table 1

$$Z\left\{\frac{2}{s+1}\right\}-Z\left\{\frac{1}{s+2}\right\}=2\frac{z}{z-e^{-1h}}-\frac{z}{z-e^{-2h}}$$

$$=\frac{2z\left(z-e^{-2h}\right)-z\left(z-e^{-h}\right)}{\left(z-e^{-1h}\right)\left(z-e^{-2h}\right)}=\frac{z^2-z\left(2e^{-2h}+e^{-h}\right)}{z^2-z\left(e^{-1h}+e^{-2h}\right)+e^{-3h}}$$

with ZOH

$$Z\left\{\frac{1-e^{-hs}}{s}\left(\frac{2}{s+1}-\frac{1}{s+2}\right)\right\} = Z\left\{\frac{1-e^{-hs}}{s}\frac{2}{s+1}\right\} - Z\left\{\frac{1-e^{-hs}}{s}\frac{1}{s+2}\right\}$$

from this point forward there are two possibilities:

a) using Table 1

$$= (1-z^{-1})Z\left\{\frac{2}{s(s+1)}\right\} - (1-z^{-1})Z\left\{\frac{1}{s(s+2)}\right\}$$

$1-e^{-at}$	$\frac{a}{s(s+a)}$	$1-e^{-anh}$	$\frac{z(1-e^{-ah})}{(z-1)(z-e^{-ah})}$

$$2\frac{z(1-e^{-1h})}{(z-1)(z-e^{-1h})} \qquad \frac{1}{2}\frac{z(1-e^{-2h})}{(z-1)(z-e^{-2h})}$$

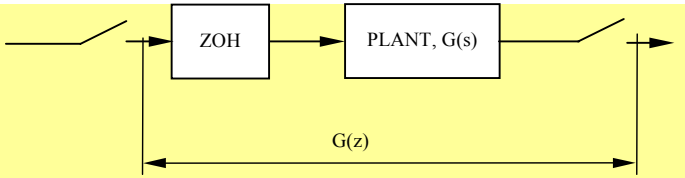
$$= \frac{z-1}{z}2\frac{z(1-e^{-1h})}{(z-1)(z-e^{-1h})} - \frac{z-1}{z}\frac{1}{2}\frac{z(1-e^{-2h})}{(z-1)(z-e^{-2h})} = 2\frac{1-e^{-1h}}{z-e^{-1h}} - \frac{1}{2}\frac{1-e^{-2h}}{z-e^{-2h}}$$

b) using Table 4

$$= Z\left\{\frac{1-e^{-hs}}{s}\frac{2}{s+1}\right\} - Z\left\{\frac{1-e^{-hs}}{s}\frac{1}{2}\frac{1}{0.5s+1}\right\}$$

$$= 2\frac{1-e^{-h/1}}{z-e^{-h/1}} - \frac{1}{2}\frac{1-e^{-h/0.5}}{z-e^{-h/0.5}}$$

$$= \frac{2\left(1-e^{-h/1}\right)\left(z-e^{-h/0.5}\right) - 0.5\left(1-e^{-h/0.5}\right)\left(z-e^{-h/1}\right)}{\left(z-e^{-h/1}\right)\left(z-e^{-h/0.5}\right)}$$



G(s)	G(z)
$\frac{1}{1+T_1 s}$	$\frac{1-e^{-\frac{h}{T_1}}}{z-e^{-\frac{h}{T_1}}}$

without ZOH it was

$$\frac{z^2 - z\left(2e^{-2h} + e^{-h}\right)}{z^2 - z\left(e^{-1h} + e^{-2h}\right) + e^{-3h}}$$