Diophantine equation: 1. Order of polynomials

Write $A(z^{-1})S(z^{-1})+z^{-d}B(z^{-1})R(z^{-1})=P(z^{-1})$ so that delay d is included in B:

$$A(z^{-1})S(z^{-1})+B(z^{-1})R(z^{-1})=P(z^{-1})$$

$$A(z^{-1})=1+a_1z^{-1}+\cdots+a_{n_A}z^{-n_A}$$

$$B(z^{-1})=b_0+b_1z^{-1}+\cdots+b_{n_B}z^{-n_B}$$

$$S(z^{-1})=s_0+s_1z^{-1}+\cdots+s_{n_S}z^{-n_S}$$

$$R(z^{-1})=r_0+r_1z^{-1}+\cdots+r_{n_R}z^{-n_R}$$

$$P(z^{-1})=p_0+p_1z^{-1}+\cdots+p_rz^{-r}$$

assuming deg A $=n_A$, deg B $=n_B$ (with delay included)

Make both AS and BR have degree r:

=> deg S = $r-n_A$, deg R = $r-n_B$

then deg P = r

=> #equations = r+1 (powers of z in P)

#unknowns = $r-n_A+1+r-n_B+1$ (coefficients in S and R)

Set #equations = #unknowns:

$r+1 = r-n_A+1+r-n_B+1$

⇔ $r = n_A+n_B-1$ (a unique solution is ensured)

=> deg S = n_B-1, deg R = n_A-1

Diophantine equation: 2. Solution via Sylvester matrix

write $A(z^{-1})S(z^{-1})+B(z^{-1})R(z^{-1})=P(z^{-1})$
as a matrix equation EM = P

$$\begin{bmatrix} a_{n_A} & & & \mathbf{0} & b_{n_B} & & & \mathbf{0} \\ a_{n_A-1} & a_{n_A} & & & b_{n_B-1} & b_{n_B} & & \\ & a_{n_A-1} & \ddots & & & b_{n_B-1} & \ddots & \\ a_1 & & \ddots & a_{n_A} & b_1 & & \ddots & b_{n_B} \\ 1 & a_1 & & a_{n_A-1} & b_0 & b_1 & & b_{n_B-1} \\ & 1 & \ddots & & & b_0 & \ddots & \\ & & \ddots & a_1 & & & \ddots & b_1 \\ \mathbf{0} & & & 1 & \mathbf{0} & & & b_0 \end{bmatrix} \begin{bmatrix} s_{n_B-1} \\ \vdots \\ s_1 \\ s_0 \\ r_{n_A-1} \\ \vdots \\ r_1 \\ r_0 \end{bmatrix} = \begin{bmatrix} p_{n_A+n_B-1} \\ \\ \\ \vdots \\ \\ \\ p_1 \\ p_0 \end{bmatrix}$$

Solving for M gives

<u>M = E⁻¹P</u>

Example: $n_A=n_B=2$ ($n_S=n_R=1$, $n_P=3$)

$$AS+BR=P$$

$$\begin{aligned} &(1+a_1z^{-1}+a_2z^{-2})(s_0+s_1z^{-1}) \\ &\quad +(b_0+b_1z^{-1}+b_2z^{-2})(r_0+r_1z^{-1}) \\ &\quad = p_0+p_1z^{-1}+p_2z^{-2}+p_3z^{-3} \end{aligned}$$

$$\begin{aligned} &s_0+a_1s_0z^{-1}+a_2s_0z^{-2} \\ &+s_1z^{-2}+a_1s_1z^{-2}+a_2s_1z^{-3} \\ &+b_0r_0+b_1r_0z^{-1}+b_2r_0z^{-2} \\ &+b_0r_1z^{-1}+b_1r_1z^{-2}+b_2r_1z^{-3} \\ &\quad = p_0+p_1z^{-1}+p_2z^{-2}+p_3z^{-3} \end{aligned}$$

$$\begin{cases} a_2s_1+b_2r_1=p_3 & \text{coeff. of } z^{-3} \\ \quad\vdots & \\ s_0+b_0r_0=p_0 & \text{coeff. of } z^{0} \end{cases}$$

Diophantine equation: 3. Design example

Process B/A and desired dynamics P_D

$$\frac{B(z^{-1})}{A(z^{-1})}=\frac{z^{-1}+2z^{-2}}{1+z^{-1}+0.5z^{-2}},P_D(z)=1$$

We have $n_A=n_B=2$, and

$$S(z^{-1})=s_0+s_1z^{-1}$$
$$R(z^{-1})=r_0+r_1z^{-1}$$

Form matrices E and P and solve M

$$\mathbf{E}=\begin{bmatrix}a_2 & 0 & b_2 & 0\\ a_1 & a_2 & b_1 & b_2\\ 1 & a_1 & b_0 & b_1\\ 0 & 1 & 0 & b_0\end{bmatrix}=\begin{bmatrix}0.5 & 0 & 2 & 0\\ 1 & 0.5 & 1 & 2\\ 1 & 1 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix},\mathbf{P}=\begin{bmatrix}p_3\\ p_2\\ p_1\\ p_0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\Rightarrow\mathbf{M}=\begin{bmatrix}s_1\\ s_0\\ r_1\\ r_0\end{bmatrix}=\mathbf{E}^{-1}\mathbf{P}=\begin{bmatrix}-1.2\\ 1\\ 0.3\\ 0.2\end{bmatrix}\qquad\Rightarrow\begin{cases}S(\mathrm{z}^{-1})=1-1.2z^{-1}\\ R(\mathrm{z}^{-1})=0.2+0.3z^{-1}\end{cases}$$