Lecture 2

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September 21, 2009

1 Arbitrage and Statistical Arbitrage

1.1 Arbitrage

Arbitrage is a financial operation where one obtains a profit with probability one by a simultaneous selling and buying of assets. A reasonable system of prices should be such that arbitrage is excluded. The no-arbitrage principle can sometimes be used to price derivatives, as in the case of futures, see Section 2.1, or as in the case of options in the Black-Scholes model. The no-arbitrage principle can also be used to give bounds to prices, or to show that the prices of two assets should be the same, see Section 2.2. We give two examples of arbitrage.

1. The stock of Nokia is listed both in Helsinki and Frankfurt stock exchanges. If the stock can be bought in Frankfurt with the price of 10 Euros and sold in Helsinki with the price of 11 Euros, we obtain a risk free profit of one Euro (minus the transaction costs).

2. Suppose the price of Nokia stock is 10 Euros and a call option with strike price $K = 8$ Euros with the expiration time in one week can be bought with the price of 1 Euro. Then we can sell the stock short and buy the call option. The profit of the operation will be $-1+10-8=1$ Euro (buying the call costs 1 Euro, selling the stock short gives 10 Euros, and excercising the option costs 8 Euros).

   In general, we have a lower bound $S_t - K$ for the price of a call option, where S_t is the price of the stock at the time of buying the option, and K is the strike price. See (4) for a more precise lower bound.

1.2 Statistical Arbitrage

Statistical arbitrage is a financial operation where one obtains a profit with high probability. The principle of excluding the possibilities of statistical arbitrage is a pricing principle which can be used sometimes when the principle of excluding arbitrage does not apply. However, the concept of statistical arbitrage is more vague than the concept of aribitrage. Let us compare the principle of excluding arbitrage to the concept of excluding statistical arbitrage.

1. The principle of excluding arbitrage works in pricing in the following way. Let us have a derivative whose value is D_T at time T. Let us have an other asset whose value is A_T at time T. Assume that the values are equal with probability one: $P(D_T = A_T) = 1$. Then it should hold that the value of the derivative and the other asset are equal at all previous times: $D_t = A_t$ for all previous times t. Otherwise, there would be an arbitrage oppurtinity: sell the more expensive instrument and buy the cheaper instrument to obtain a risk free profit at time T.

2. The principle of excluding statistical arbitrage works in pricing in the following way. Let us have a derivative whose value is D_T at time T. Let us have an other asset whose value is A_T at time T. If the random varibles D_T and A_T are "close", then the prices D_t and A_t should be close at all previous times t. The closeness of random variables can be defined in many ways. For example, we can say that two random variables D_T and A_T are close when $E(D_T - A_T)^2$ is small. A derivative can be priced with statistical arbitrage if we can construct an asset which replicates the payoff of the derivative with high probability.

We give examples of attempts of using statistical arbitrage in pricing.

1. Consider a coin-tossing game where one obtains 1 Euro when heads occurs and 0 Euros when tails occurs. The probability of obtaining heads is 0.5 and the probability of obtaining tails is 0.5. What is the fair price of this lottery?

   The fair price can be considered to be the expected gain:

   $$0.5 \cdot 1 \text{ Euros} + 0.5 \cdot 0 \text{ Euros} = 0.5 \text{ Euros}.$$

   The fairness of the price can be justified by the following arguments.

   (a) By the law of large numbers the gain from repeated independent repetitions of the game with price 0.5 Euros converges to zero with probabilty one (a larger price than 0.5 Euros would give an almost sure profit to the organizer of the game in the long run and a smaller price than 0.5 Euros would give an almost sure profit to the player of the game in the long run).

(b) The best constant approximation in the sense of the L_2 error of a square integrable random variable X is its expectation:

$$E(X-P)^2 = E(X-EX)^2 + (EX-P)^2$$

and thus

$$\text{argmin}_{P\in\mathbf{R}} E(X-P)^2 = EX,$$

where the minimization is taken with respect to all real numbers.

Note that the situation changes when it costs 0.5 million Euros to buy a ticket and the payoff is 1 million Euros when heads occur. Then the probability of a bankrupcy increases so much that the strategy of repetitions is unfeasible. Note also that a doubling strategy gives an almost sure win. Following the classical doubling strategy, the player doubles his bet until the first time he wins. If he starts with 1 Euro, his final gain is 1 Euro almost surely.

2. *The St. Petersburg Paradox* The banker flips the coin until the heads come out the first time. The player receives 2^{k-1} coins when there are k tosses of the coin (1 coin if the heads come out in the first toss, 2 coins if the heads come out in the second toss, 4 coins if the heads come out in the third toss, and so on). What is the fair entrance fee to the game?

We can calculate the expected gain. The probability that there are k tosses is $p_k = 2^{-k}$. Thus the expeted payoff is

$$\sum_{k=1}^{\infty} p_k 2^{k-1} = \sum_{k=1}^{\infty} \frac{1}{2} = \infty.$$

Thus it would seem that the entrance fee could be arbitrarily high. However, applying common sense, it does not seem reasonable to pay a high entrance fee. The solution of the paradox is to use a utility function to measure the utility of the wealth. For example, the logarithmic utility function $x \mapsto \log_e(x)$ gives the expected utility of the game

$$\sum_{k=1}^{\infty} p_k \log_e(2^{k-1}) = \log_e 2,$$

which would give the price of 2 coins for the game.

The St. Petersburg paradox shows that in general one should use the expected utility instead of expected monetary payoff in pricing. However, it is not clear which utility function we should use, since the choice of the utility function depends on the individual risk aversion. Also, the main need for a logarithmic utility seems to be come from the cases of extreme returns and extreme probabilities and locally we can approximate a logarithmic function with a linear function. Thus we will not use utility functions in asset pricing.

3. *Call Option* Consider a call option written at time t, with the strike price K, with the expiration time T, when the current price of the underlying stock is S_t. The payout of the option at the expiration time T is $C_T = \max\{S_T - K, 0\}$, but what would be the fair price of the option at time t?

   There are at least three differences in the setting of option pricing, as compared to the pricing of coin flipping games. (1) In the coin flipping games the payment is made and the payoff is received almost simultaneously, whereas the expiration time of the option can be several months or even years ahead. Thus one has to take into account the cost of money in option pricing. (2) The probabilities of the outcomes are known in the coin flipping, whereas the distribution of stock prices is unknown and has to be estimated with statistical techniques. [1] (3) In the coin flipping we cannot invest to any games whose outcome is related to the game we are pricing, whereas in the case of options we can make transactions with the underlying stock.

   Let us denote by C_t the price of the option at time t. The writer of the option collects C_t Euros and puts this money to a bank account, obtaining $e^{r(T-t)}C_t$ Euros at time T.[2] Thus the wealth of the writer of the option at time T is $e^{r(T-t)}C_t - C_T$. The buyer of the option borrows C_t Euros in order to be able to buy the option, and thus the buyers wealth at time T is $C_T - e^{r(T-t)}C_t$. We could take the fair price of the

[1]Arbitrage is based on "known knowns", statistical arbitrage in the coin flipping is based on "known unknowns", whereas statistical arbitrage in option pricing has to deal with "unknown unknowns".

[2]We use the methof of continuous combounding. The interest rate for the period from t to T is denoted by $r > 0$. Assume that there are n payments of interest. Every payment increases the savings by the factor $(1 + r \cdot (T-t)/n)$. Then the total combounded savings at time T is $(1 + r \cdot (T-t)/n))^n P \to e^{r(T-t)}P$ as $n \to \infty$, where $P > 0$ is the initial savings.

option to be the solution of

$$E_t\left(C_T - e^{r(T-t)}C_t\right) = 0$$

which gives

$$C_t = e^{-r(T-t)}E_tC_T. \tag{1}$$

Here the expectation is taken with respect to the distribution of S_T given the price S_t, and this expectation operator is denoted by E_t.

It turns out that the price in (1) still allows statistical arbitrage and cannot be considered fair. Indeed, at time t and at all times before the expiration time T it is possible to make operations with the underlying stock. (In contrast to the coin tossing examples, where we do not have access to any games whose outcome is related to the outcome of the original game.) The writer of the option can buy a fraction of the option and hedge his position. The fair price is somewhat lower than (1), thanks to the writer's possibility of hedging his position.

2 Pricing by Arbitrage

The no-arbitrage principle can be used to give a unique price to a derivative. The idea is that if two assets have the same price at the end of the period, in every state of the world, then the assets should have the same price at the beginning of the period. Otherwise, we could obtain a risk free profit by selling the more expensive asset and by buying the cheaper asset. We give several examples where arbirage pricing can be applied.

2.1 Futures

2.1.1 Futures on a Stock

We consider a futures contract on a stock. The futures contract is made at time t and the contract specifies that the buyer of the contract has to buy the stock at a later time T with price K. We assume that the stock does not pay dividends during the time period from t to T. Let us denote with S_t and S_T the prices of the stock at times t and T. The value of the futures contract at time T is

$$F_T = S_T - K,$$

because the buyer of the futures contract gives away K and receives S_T. We want to determine the fair value F_t at time t of the futures contract. However, since the futures contract is such that nothing changes hands at time t, the problem transforms into choosing the fair value for K, which is such that the value of the futures contract is zero at time t: $F_t = 0$. (The pricing problem may stated as finding such fair forward price K that value F_t is zero.) We may replicate the futures contract buy buying the stock S_t and borrowing the amount $e^{-r(T-t)}K$, where $r > 0$ is the interest rate for the period from t to T. At time t the value of this portfolio is

$$S_t - e^{-r(T-t)}K.$$

One can see immediately that at time T the value of this portfolio is $F_T = S_T - K$ with probability one. Thus,

$$F_t = S_t - e^{-r(T-t)}K,$$

to exclude arbitrage. [3] Choosing $F_t = 0$ gives

$$K = K_t = e^{r(T-t)}S_t. \tag{2}$$

When an investor enters a futures contract in a futures exchange, this does not imply any cash flows, but the exchange requires from the investor a liquid collateral in order to secure a possible future payment. The future prices which are quoted in a futures exchange are the forward prices K_t, and these are determined in the end by the supply and demand. Numbers K_t are called futures prices or forward prices.

A buyer of a stock future or a stock index future saves the carrying costs but looses the possible stock dividends. When the yearly dividend rate is knwown to be d, then the fair future price is

$$K_t = e^{(r-d)(T-t)}S_t.$$

2.2 Put-Call Parity

The price of a put can always be expressed in terms of the price of a call, and conversely. We have the put-call parity:

$$C_t - P_t = S_t - Ke^{-r(T-t)}, \tag{3}$$

[3] Let us denote $B_t = e^{-r(T-t)}$ a risk free bank account, where $r > 0$ is the rate for the period from t to T. We replicate the futures contract with the portfolio

$$V_t = \phi_t S_t + \psi_t B_t,$$

where $\phi_t \equiv 1$ and $\psi_t = -e^{r(T-t)}K_t$. This portfolio has always the same value at the expiration date T as the futures contract: $V_T = F_T$. Thus the value of the futures contract at time t is

$$F_t = V_t = S_t - e^{-r(T-t)}K_t.$$

where K is the common strike price of the call and put, and r is the interest rate for the period from t to T. It is clear that at the expiration we have $C_T - P_T = S_T - K$. The put-call parity extends this result for times t before the expiration time T. We do not need to know fair values for C_t and P_t in order to have a formula for their difference.

2.2.1 Derivation of the Put-Call Parity

Consider the portfolio $V_t^{(1)}$ where we buy the call and sell the put:

$$V_t^{(1)} = C_t - P_t.$$

At the expiration we have $V_T^{(1)} = S_T - K$. This can be seen immediately:

1. If $S_T \leq K$, then the call option expires worthless ($C_T = 0$), and the value of the put option is $P_T = K - S_T$. Thus in this case $C_T - P_T = S_T - K$.

2. If $S_T \geq K$, then the value of the call option is $C_T = S_T - K$ and the put option is worthless ($P_T = 0$). Thus also in this case $C_T - P_T = S_T - K$.

Consider the portfolio $V_t^{(2)}$ where we buy the stock and borrow the amount $Ke^{-r(T-t)}$:

$$V_t^{(2)} = S_t - Ke^{-r(T-t)}.$$

At the expiration we have $V_T^{(2)} = S_T - K$. Since with probability 1, $V_T^{(1)} = V_T^{(2)}$, we have

$$V_t^{(1)} = V_t^{(2)}$$

for all times t before T, to exclude arbitrage. This is equivalent to (3).

2.2.2 Consequencies of the Put-Call Parity

Bounds for the option price We have that

$$\max\left\{S_t - e^{-r(T-t)}K, 0\right\} \leq C_t \leq S_t. \tag{4}$$

Indeed, $C_t \geq 0$ is obvious, since the right to buy a stock involves no obligations. Also, $C_t \leq S_t$ is obvious, since the right to buy a stock must be less valuable than than the stock itself. The put call parity and the fact that $P_t \geq 0$ gives

$$S_t - Ke^{-r(T-t)} = C_t - P_t \leq C_t.$$

Valuation of Options We can derive a value for the calls and puts using the put-call parity. We make the asumption: The value of the call option is equal to

$$e^{-r(T-t)}EC_T = e^{-r(T-t)}E(S_T - K)_+,$$

where the expectation is taken with respect to the distrbution of S_T, which is taken to be

$$S_T = S_t \exp\{Z\sigma\sqrt{T-t} + \mu(T-t)\},$$

where $Z \sim N(0,1)$, $\mu \in \mathbf{R}$, and $\sigma > 0$. Now, denoting $\phi(z) = (2\pi)^{-1/2}e^{-z^2/2}$, $z \in \mathbf{R}$, the density of standard the Gaussian distribution,

$$E(S_T - K)_+ = \int_w^\infty \left(S_t \exp\{z\sigma\sqrt{T-t} + \mu(T-t)\} - K\right)\phi(z)\,dz,$$

where

$$w = \frac{\log_e(K/S_t) - \mu(T-t)}{\sigma\sqrt{T-t}}.$$

We have

$$\exp\left\{z\sigma\sqrt{T-t}\right\}\phi(z) = \exp\left\{\frac{1}{2}\sigma^2(T-t)\right\}\phi\left(z - \sigma\sqrt{T-t}\right).$$

Thus,

$$\begin{aligned}
E(S_T - K)_+ &= S_t e^{\mu(T-t)} \int_w^\infty e^{z\sigma\sqrt{T-t}}\,\phi(z)\,dz - K\int_w^\infty \phi(z)\,dz \\
&= S_t e^{\mu(T-t)+\sigma^2(T-t)/2} \int_{w-\sigma\sqrt{T-t}}^\infty \phi(z)\,dz - K\int_w^\infty \phi(z)\,dz \\
&= S_t e^{\mu(T-t)+\sigma^2(T-t)/2}\Phi(\sigma\sqrt{T-t} - w) - K\Phi(-w).
\end{aligned}$$

The value of the put option is equal to

$$e^{-r(T-t)}EP_T = e^{-r(T-t)}E(K - S_T)_+$$

and

$$\begin{aligned}
E(K - S_T)_+ &= \int_{-\infty}^w \left(K - S_t \exp\{z\sigma\sqrt{T-t} + \mu(T-t)\}\right)\phi(z)\,dz \\
&= K\Phi(w) - S_t e^{\mu(T-t)+\sigma^2(T-t)/2}\Phi(w - \sigma\sqrt{T-t}).
\end{aligned}$$

Thus

$$e^{-r(T-t)}EC_T - e^{-r(T-t)}EP_T = S_t e^{(\mu+\sigma^2/2-r)(T-t)} - e^{-r(T-t)}K,$$

because $\Phi(x)+\Phi(-x) = 1$ for all $x \in \mathbf{R}$. The put-call parity (3) implies that we have to take

$$\mu = r - \frac{1}{2}\sigma^2.$$

Then we can write the value of the call option as

$$e^{-r(T-t)} EC_T = S_t \Phi(z_+) - K e^{-r(T-t)} \Phi(z_-),$$

where

$$z_{\pm} = \frac{\log_e(S_t/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

which is known as the Black-Scholes price of the call option. This derivation was noted in Derman and Taleb (2005).

3 Examination

Possible questions in the examination:

3) Derive the arbitrage free forward price for a futures contract.

4) Derive the put-call parity.

References

Derman, E. and Taleb, N. N. (2005), 'The illusions of dynamic replication', *Wilmott Magazine* .