

Lecture 4

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1 Risk Adjusted Return

Univariate statistics can be used to describe statistical properties of a portfolio selection method. The univariate statistics can be divided into measures of return and to measures of risk. The final ranking of portfolio selection methods is made by balancing return and risk. The statistics which balance return and risk are called risk adjusted return statistics.

We assume to have a time series of returns

$$Y_t = \frac{W_t}{W_{t-1}} - 1, \quad t = 1, \dots, T,$$

where W_0, \dots, W_T is a wealth process of the portfolio. The time series is typically either a time series of daily returns or a time series of monthly returns. This time series is used to estimate the characteristics of a portfolio selection method.

1.1 Return and Risk of a Portfolio

Return The return of a portfolio is typically estimated using arithmetic mean and it is expressed typically as annualized return:

$$250 \times \bar{Y}_T, \quad \bar{Y}_T = \frac{1}{T} \sum_{t=1}^T Y_t,$$

where the annualization multiplier 250 is chosen in the case of daily returns because there are about 250 trading days in a year. In the case of monthly returns the annualization multiplier 12 is used. The geometric mean of the returns is calculated using the formula

$$250 \times \bar{Y}_{g,T}, \quad \bar{Y}_{g,T} = \left(\prod_{t=1}^T (Y_t + 1) \right)^{1/T} - 1.$$

Risk The volatility of the portfolio is expressed typically as the annualized standard deviation:

$$\sqrt{250} \times \widehat{\text{std}}(Y), \quad \widehat{\text{std}}(Y) = \left(\frac{1}{T} \sum_{t=1}^T (Y_t - \bar{Y}_T)^2 \right)^{1/2},$$

in the case of daily returns, and in the case of monthly returns the multiplier $\sqrt{12}$ is used. The volatility is not a satisfactory risk measure because it takes equally into account the losses and gains. However, we are interested in the left tail of the return distribution when we want to measure the riskiness of a portfolio. Also, volatility is not suited to measure the risk arising from low probability events. We can use the following alternatives to the volatility:

1. The sample lower partial moments measure the risk of losses. The second lower partial moment is called also the downside volatility.
2. The sample negative quantile $-\widehat{Q}_p(Y)$, measures the risk of losses when $0 < p < 0.5$.
3. We can also use the shortfall or absolute shortfall to measure risks.

Drawdown Drawdown is a new time series constructed from the time series W_0, W_1, \dots, W_T of wealths. This time series is defined as

$$D_t = W_{t,\max} - W_t, \quad t = 0, 1, \dots, T,$$

where

$$W_{t,\max} = \max\{W_i : i = 0, \dots, t\}.$$

The interesting statistics are the maximum drawdown, the mean drawdown, and the variance of drawdowns.

1.2 Measures of Risk Adjusted Return

We can increase as much as we like the expected return of a given portfolio by leveraging. However, leveraging increases also the risk of the portfolio. Consider a portfolio with expected return $r > 0$ and the variance of the returns $\sigma^2 > 0$. Let the initial wealth be W_0 and the wealth at the next step be W_1 . Denote the wealth ratio by $U = W_1/W_0$. By assumption,

$$E(U) = 1 + r, \quad \text{Var}(U) = \sigma^2.$$

We borrow amount kW_0 , where $k > 0$, and invest this amount in addition to the initial wealth W_0 . We have to pay back $(1 + \rho)kW_0$, where $\rho > 0$ is the

risk free interest rate. The wealth of the leveraged portfolio at the next step is

$$W'_1 = (W_0 + kW_0)U - (1 + \rho)kW_0$$

and thus, denoting

$$U' = \frac{W'_1}{W_0} = (1 + k)U - (1 + \rho)k,$$

we have

$$EU' = (1 + k)(1 + r) - k(1 + \rho) = 1 + r + k(r - \rho)$$

and

$$\text{Var}(U') = (1 + k)^2\sigma^2.$$

Thus the expected return increases by adding the amount $k(r - \rho)$ and the variance increases by the multiplier $(1 + k)^2$. Thus in portfolio evaluation we have to take into account also the risk of a portfolio and not only the return of a portfolio.

Sharpe Ratio The Sharpe ratio is the adjusted return divided by the standard deviation:

$$\frac{\bar{Y}_T - r}{\widehat{\text{std}}(Y)},$$

where the adjusting rate r is the risk free rate for the chosen period of time. The annualized Sharpe ratio is obtained by multiplying with the factor $\sqrt{250}$ or with the factor $\sqrt{12}$, depending on whether we have used daily or monthly returns. Sharpe ratio was defined by Sharpe (1966).

Sortino ratio The Sortino ratio is just like the Sharpe ratio, but the denominator is the downside volatility:

$$\frac{\bar{Y}_T - r_{mar}}{\widehat{\text{std}}_d(Y)},$$

where the downside volatility is defined $\widehat{\text{std}}_d(Y) = \widehat{\text{LPM}}_{r_{mar},2}(Y)$ with the lower partial moment, and the rate $r_{mar} > 0$ is the same threshold as used in the calculation of the downside volatility.

Omega Ratio We have defined the upper and lower partial moments. The population Omega ratio is the ratio of the upper partial moment of order one to the lower partial moment of order one:

$$\Omega(r) = \frac{E(Y - r)_+}{E(r - Y)_+},$$

where r is the chosen threshold. The definition was made in Shadwick and Keating (2002). Note that the Omega ratio is written often using the expressions

$$E(Y - r)_+ = \int_r^\infty (1 - F_Y(y)) dy$$

and

$$E(r - Y)_+ = \int_{-\infty}^r F_Y(y) dy,$$

where $F_Y : \mathbf{R} \rightarrow \mathbf{R}$ is the distribution function of Y . The sample Omega ratio is

$$\frac{\sum_{i=1}^T (Y_i - r)_+}{\sum_{i=1}^T (r - Y_i)_+}.$$

1.2.1 Markowitz Approach

Portfolio choice with mean-variance preferences was proposed by Markowitz (1952) and Markowitz (1959). This method ranks the distributions according to

$$EY - \frac{\gamma}{2} \text{Var}(Y),$$

where $\gamma \geq 0$ is the coefficient of absolute risk aversion. Parameter γ measures the investor's absolute risk aversion. Remember that $Y \sim Y_t$, where $Y_t = W_t/W_{t-1} - 1$, $W_t = W_{t-1} \cdot b_{t-1}^T U_t$, and $U_t = (S_t^1/S_{t-1}^1, \dots, S_t^d/S_{t-1}^d)$. Thus $Y_t = b_{t-1}^T U_t - 1$. Let us write $b = b_{t-1}$, $U \sim U_t$, so that $Y = b^T U - 1$. We can write

$$EY = b^T EU - 1, \quad \text{Var}(Y) = b^T \text{Var}(U)b,$$

where $\text{Var}(U)$ is the $d \times d$ covariance matrix of U . Thus we have to estimate the vector expected returns EU , and the covariance matrix $\text{Var}(U)$, which can be done using the historical data S_0, \dots, S_{t_0} , which gives the historical returns U_t , $t = 1, \dots, t_0$. We can estimate the population moments with the sample versions and to use the arithmetic mean to estimate the expectation and the empirical covariance matrix to estimate the covariance matrix.

Let us calculate some examples of Markowitz optimal portfolio vectors.

1. Let us invest the proportion b to a stock and $1 - b$ to the risk free rate $\mu_r > 0$. Now

$$Y = b(U - 1) + (1 - b)\mu_r.$$

Let the expected return of the stock be $EU = 1 + \mu$ and the variance $\text{Var}(U) = \sigma^2$. Then,

$$EY - \frac{\gamma}{2} \text{Var}(Y) = \mu_r + b(\mu - \mu_r) - \frac{\gamma}{2} b^2 \sigma^2.$$

Setting the derivative with respect to b to zero and solving for b gives

$$b = \frac{1}{\gamma} \frac{\mu - \mu_r}{\sigma^2}.$$

2. Let us have two stocks and put the proportion $1 - b$ to the first stock and proportion b to the second stock. Now

$$Y = (1 - b)(U^1 - 1) + b(U^2 - 1).$$

Let the expected returns of the stocks be $EU^1 = 1 + \mu_1$, $EU^2 = 1 + \mu_2$ and the variances of the returns $\text{Var}(U^1) = \sigma_1^2$, $\text{Var}(U^2) = \sigma_2^2$. Denote the covariance of the returns by $\text{Cov}(U^1, U^2) = \sigma_{12}$. We have

$$\begin{aligned} EY - \frac{\gamma}{2} \text{Var}(Y) &= \mu_1 + b(\mu_2 - \mu_1) - \frac{\gamma}{2} [(1 - b)^2 \sigma_1^2 + b^2 \sigma_2^2 + 2(1 - b)b\sigma_{12}] \\ &= \mu_1 - \frac{\gamma}{2} \sigma_1^2 + b [\mu_2 - \mu_1 - \gamma(\sigma_{12} - \sigma_1^2)] - b^2 \frac{\gamma}{2} (\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}). \end{aligned}$$

Setting the derivative with respect to b to zero and solving for b gives

$$b = \frac{1}{\gamma} \frac{\mu_2 - \mu_1 - \gamma(\sigma_{12} - \sigma_1^2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.$$

3. Let us have two stocks and the risk free rate $\mu_r > 0$. Let us put the proportion b_1 to the first stock, proportion b_2 to the second stock, and proportion $1 - b_1 - b_2$ to the risk free rate. Now

$$Y = b_1(U^1 - 1) + b_2(U^2 - 1) + (1 - b_1 - b_2)\mu_r.$$

Let the expected returns of the stocks be $EU^1 = 1 + \mu_1$, $EU^2 = 1 + \mu_2$ and the variances of the returns $\text{Var}(U^1) = \sigma_1^2$, $\text{Var}(U^2) = \sigma_2^2$. Denote the covariance of the returns by $\text{Cov}(U^1, U^2) = \sigma_{12}$. We have

$$\begin{aligned} EY - \frac{\gamma}{2} \text{Var}(Y) &= b_1\mu_1 + b_2\mu_2 + (1 - b_1 - b_2)\mu_r - \frac{\gamma}{2} (b_1^2\sigma_1^2 + b_2^2\sigma_2^2 + 2b_1b_2\sigma_{12}). \end{aligned}$$

Setting derivatives with respect to b_1 and b_2 to zero gives

$$\begin{cases} \mu_1 - \mu_r - \gamma\sigma_1^2 b_1 - \gamma\sigma_{12} b_2 = 0 \\ \mu_2 - \mu_r - \gamma\sigma_2^2 b_2 - \gamma\sigma_{12} b_1 = 0. \end{cases}$$

Solving b_2 gives

$$b_2 = \frac{\mu_2 - \mu_r - \gamma\sigma_{12} b_1}{\gamma\sigma_2^2}.$$

Which gives

$$\mu_1 - \mu_r - \frac{\sigma_{12}}{\sigma_2^2} (\mu_2 - \mu_r) + b_1 \gamma \left(\frac{\sigma_{12}^2}{\sigma_2^2} - \sigma_1^2 \right) = 0.$$

Thus,

$$b_1 = \frac{1}{\gamma} \frac{(\mu_1 - \mu_r)\sigma_2^2 - \sigma_{12}(\mu_2 - \mu_r)}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$$

and

$$b_2 = \frac{1}{\gamma} \frac{(\mu_2 - \mu_r)\sigma_1^2 - \sigma_{12}(\mu_1 - \mu_r)}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}.$$

Minimizing Variance when Return is Fixed We consider now the approach where we minimize the variance of the portfolio returns under a constraint on the return.

1. Let us denote $\mu = EY$ and $\Sigma = \text{Cov}(Y)$. Then

$$E(b^T Y) = b^T \mu, \quad \text{Cov}(b^T Y) = b^T \Sigma b.$$

Let us consider the problem

$$b_0 = \text{argmin}\{b^T \Sigma b : b^T 1_d = 1, b^T \mu = \mu_0\},$$

where 1_d is the d -dimensional column vector whose elements are equal to 1. Form the Lagrangian

$$L(b, \lambda_1, \lambda_2) = b^T \Sigma b + \lambda_1(\mu_0 - b^T \mu) + \lambda_2(1 - b^T 1_d),$$

where $\lambda_1, \lambda_2 \in \mathbf{R}$ are the Lagrange multipliers. We need to solve the equation¹

$$0 = \frac{\partial}{\partial b} L(b, \lambda_1, \lambda_2) = 2\Sigma b - \lambda_1 \mu - \lambda_2 1_d.$$

¹We have that $\frac{\partial}{\partial b} b^T \Sigma b = (\Sigma + \Sigma^T)b$, but covariance matrix Σ is symmetric.

We get

$$b_0 = \frac{1}{2} \Sigma^{-1} (\lambda_1 \mu + \lambda_2 1_d).$$

To get λ_1 and λ_2 we need to solve the equations

$$\begin{cases} \mu_0 = b_0^T \mu = \frac{1}{2} (\lambda_1 \mu^T \Sigma^{-1} \mu + \lambda_2 \mu^T \Sigma^{-1} 1_d), \\ 1 = b_0^T 1_d = \frac{1}{2} (\lambda_1 1_d^T \Sigma^{-1} \mu + \lambda_2 1_d^T \Sigma^{-1} 1_d). \end{cases}$$

Denoting $a = \mu^T \Sigma^{-1} \mu$, $b = 1_d^T \Sigma^{-1} 1_d$, and $c = 1_d^T \Sigma^{-1} \mu$, we get

$$\lambda_1 = 2 \frac{-c + b\mu_0}{e}, \quad \lambda_2 = 2 \frac{a - c\mu_0}{e},$$

where $e = ab - c^2$. Then

$$b_0 = \frac{1}{e} \Sigma^{-1} [(a1_d - c\mu) + \mu_0 (b\mu - c1_d)].$$

2. Let us drop the constraint that $b^T 1_d = 1$, but assume that the proportion $1 - b^T 1_d$ is invested in the risk-free asset. The expected return is

$$b^T \mu + (1 - b^T 1_d) \mu_r,$$

where $\mu_r > 0$ is the return of the risk-free asset. Let us consider the problem

$$b_0 = \operatorname{argmin}\{b^T \Sigma b : b^T \mu + (1 - b^T 1_d) \mu_r = \mu_0\}.$$

Form the Lagrangian

$$L(b, \lambda) = b^T \Sigma b + \lambda [\mu_0 - b^T \mu - (1 - b^T 1_d) \mu_r],$$

where $\lambda \in \mathbf{R}$ is the Lagrange multiplier. We need to solve the equation

$$0 = \frac{\partial}{\partial b} L(b, \lambda) = 2\Sigma b + \lambda(-\mu + 1_d \mu_r).$$

We get

$$b_0 = \lambda \frac{1}{2} \Sigma^{-1} (\mu - 1_d \mu_r).$$

The constraint can be written as

$$b_0^T (\mu - \mu_r 1_d) = \mu_0 - \mu_r$$

which implies

$$\lambda \frac{1}{2} (\mu - \mu_r 1_d)^T \Sigma^{-1} (\mu - \mu_r 1_d) = \mu_0 - \mu_r$$

and

$$\lambda = 2 \frac{\mu_0 - \mu_r}{(\mu - \mu_r \mathbf{1}_d)^T \Sigma^{-1} (\mu - \mu_r \mathbf{1}_d)}.$$

Thus

$$b_0 = \frac{\mu_0 - \mu_r}{(\mu - \mu_r \mathbf{1}_d)^T \Sigma^{-1} (\mu - \mu_r \mathbf{1}_d)} \Sigma^{-1} (\mu - \mathbf{1}_d \mu_r).$$

2 Examination

Possible questions in the examination:

- 7) Let us have two stocks and the risk free rate. Let us put the proportion b_1 to the first stock S^1 , proportion b_2 to the second stock S^2 , and proportion $1 - b_1 - b_2$ to the risk free rate $\mu_r > 0$. Let

$$Y = b_1(U^1 - 1) + b_2(U^2 - 1) + (1 - b_1 - b_2)\mu_r,$$

where $U^1 = S_t^1/S_{t-1}^1$, $U^2 = S_t^2/S_{t-1}^2$. Find b that minimizes $EY - \frac{\gamma}{2} \text{Var}(Y)$, where $\gamma > 0$.

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