Lecture 6

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1 Univariate Models

1.1 Normal and Log-normal Models

The two classical models for the distribution of the stock prices are the normal and the log-normal model. The normal model assumes that the increments of the stock prices are Gaussian, whereas the log-normal model assumes that the increments of the logarithms of the stock prices are Gaussian. The Gaussian model goes back to Bachelier (1900) and the Black-Scholes pricing is based on the continuous time limit of the log-normal model.

Heuristics of the Models We define the models using the heuristics of the central limit theorem.

1. *Normal Model* Let S_t be the price of an asset, $t = 0, \ldots, T$. We may write the price at time T as

   $$S_T = S_0 + \sum_{t=0}^{T-1} (S_{t+1} - S_t).$$

   If the price increments $S_{t+1} - S_t$ are i.i.d. with expectation μ and variance σ^2, then an application of the central limit theorem gives the approximation

   $$S_T \sim N\left(T\mu + S_0, T\sigma^2\right). \tag{1}$$

   Equation (1) defines the Gaussian model for the asset prices. We can write the log-normal model also

   $$S_T = S_0 + T\mu + \sqrt{T}\,\sigma Z,$$

   where $Z \sim N(0,1)$ is a random variably which has standard normal distribution.

2. *Log-normal Model* We may write the asset price at time T as

$$S_T = S_0 \cdot \prod_{t=0}^{T-1} \frac{S_{t+1}}{S_t} = S_0 \cdot \exp\left\{ \sum_{t=0}^{T-1} \log_e \left(\frac{S_{t+1}}{S_t} \right) \right\}.$$

If the logarithms of the price ratios are i.i.d. with expectation m and variance s^2, then an application of the central limit theorem gives the approximation

$$\log S_T \sim N\left(Tm + \log_e S_0, Ts^2\right). \qquad (2)$$

This is equivalent to saying that S_T is log-normally distributed with parameters $Tm + \log_e S_0$ and $\sqrt{T}s$. Equation (2) defines the log-normal model for the asset prices. We can write the log-normal model also

$$S_T = S_0 \exp\left\{ Zs\sqrt{T} + mT \right\},$$

where $Z \sim N(0,1)$ is a random variably which has standard normal distribution. The log-normal model was applied in Lecture 2 to derive a price for options.

The moments of a log-normal variable are $x_0^k e^{k^2\sigma^2/2}$, where $x_0 = e^\mu$. The log-normal density is

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(\log_e t - \mu)^2}{2\sigma^2} \right\}, \qquad t > 0.$$

Volatility The parameter σ in (1) or the parameter s in (2) are called the volatility parameter. The volatility is typically quoted as the annualized volatility, which is equal to

$$\sqrt{250} \cdot s \qquad (3)$$

or $\sqrt{250} \cdot \sigma$. The actual number of trading days in a year is between 250 and 252. We have assumed that the prices $S_0, \ldots, S_T$ are daily prices. If the prices are recorded with a different frequency, making k records in the day, then the annualized volatility is equal to

$$\sqrt{250/k} \cdot s.$$

The volatility can be estimated in the log-normal model as the square root of

$$\hat{s}^2 = \frac{1}{T} \sum_{t=0}^{T-1} X_t^2 - \left(\frac{1}{T} \sum_{t=0}^{T-1} X_t \right)^2,$$

where $X_t = \log_e(S_{t+1}/S_t)$. In the Gaussian model we take $X_t = S_{t+1} - S_t$.

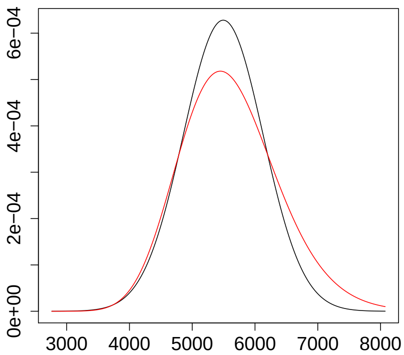

Figure 1: *Normal and Log-normal Densities* The normal density has black graph and the log-normal has red graph. The densities are estimates of the distribution of the DAX stock price 90 days to the future.

Density Functions Figure 1 shows estimated distributions of DAX stock index in 90 days from 2009-08-10. The closing price of the DAX is 5418.12 at 2009-08-10. We have the daily clsoing prices of DAX starting at 1990-11-26, which makes 4722 observations. The estimates for μ and σ (normal model) are 0.84 and 67.0. The estimates for m and s (log-normal model) are 0.000028 and 0.015. The log-normal density is shown red and it is skewed to the left and has heavier right tail than the left tail. The normal density is shown black and it is symmetric with respect to the mean.

Remarks We make two remarks concerning the normal and log-normal distributions.

1. Log-normal distribution takes only positive values. Gaussian distributions can take negative values but the tail of the distribution is so thin that the probability of negative values can be very small. Thus the positivity of log-normal distributions is not a strong argument in favor of its use to model prices. (Gaussian distributions are used in any case as an approximation.)

2. In the Black-Scholes model 105 call has more value than 95 put when the stock is at 100. This can be explained by the fact that the log-normal distribution allows for greater upside price movements than downside price movements.

1.2 Heavy Tailed Distributions

A distribution of random variable $X \in \mathbf{R}$ with distribution function $F_X : \mathbf{R} \to [0,1]$ is said to have a Pareto right tail when

$$P(X \geq x) = 1 - F_X(x) = L(x)\, x^{-\alpha}, \tag{4}$$

for $x > 0$, for some $\alpha > 0$, where $L : (0,\infty) \to (0,\infty)$ is a slowly varying function at $+\infty$:

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1,$$

for all $\lambda > 0$.[1] A distribution is said to have a Pareto left tail when

$$P(X \leq -x) = F_X(-x) = L(x)\, x^{-\alpha}, \tag{5}$$

for $x > 0$, for some $\alpha > 0$, where $L : (0,\infty) \to (0,\infty)$ is a slowly varying function. If density $f : \mathbf{R} \to \mathbf{R}$ satisfies

$$f(x) = Ax^{-1-\alpha}$$

for large and small x, where $A > 0$, then the distribution has Pareto tails.

2 Estimation of the Tails

2.1 Tail Index

Semiparametric Method Let us consider the estimation of the tail index α under the assumption of the Pareto tail of the distribution. Let us consider the estimation of the left tail and make the assumption (5). Under this assumption we have

$$\log P(Y \leq -y) = \log L(y) - \alpha \log(y),$$

where $y > 0$. If y is sufficiently large, we can approximate $L(y)$ with a constant: $L(y) \approx L$, where $L > 0$. Let $Y_{(1)} < \cdots < Y_{(T)}$ be the order statistics. Then,

$$P(Y \leq Y_{(k)}) \approx k/T.$$

Thus

$$\log(k/T) \approx \log L - \alpha \log(-Y_{(k)}).$$

[1] If $0 < \lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} < \infty$ for all λ, then $L : (0,\infty) \to (0,\infty)$ is called regularly varying.

We can consider this as a problem of linear regression where the values of the explanatory variable are $X_k = \log(-Y_{(k)})$ and the values of the response variable are $Z_k = \log(k/T)$. The linear regression is performed on the $5\% - 20\%$ of the observations. Thus we have regression data (X_k, Z_k), $k = 1, \ldots, T_0$, where $T_0 = [pT]$, and p is something between 0.05 and 0.20. The least squares estimate is

$$\hat{\alpha} = \frac{\sum_{k=1}^{T_0}(X_k - \bar{X})(Z_k - \bar{Z})}{\sum_{k=1}^{T_0}(X_k - \bar{X})^2}, \qquad \widehat{\log L} = \bar{Z} - \hat{\alpha}\bar{X},$$

where

$$\bar{X} = \frac{1}{T_0}\sum_{k=1}^{T_0} X_k, \qquad \bar{Z} = \frac{1}{T_0}\sum_{k=1}^{T_0} Z_k.$$

Hill's Estimator The Pareto distribution has distribution function

$$F(x) = 1 - \left(\frac{c}{x}\right)^{\alpha}, \qquad x > c,$$

and density function

$$f(x) = \frac{\alpha c^{\alpha}}{x^{\alpha+1}}, \qquad x > c.$$

This is a model for the right tail. Assume that c is known. Define

$$\mathrm{Lik}(\alpha) = \prod_{i=1}^{T} \frac{\alpha c^{\alpha}}{Y_i^{\alpha+1}}.$$

Taking logarithms

$$\log \mathrm{Lik}(\alpha) = \sum_{i=1}^{T} \left[\log(\alpha) + a\log(c) - (\alpha+1)\log Y_i\right].$$

Differentiating with respect to α and setting the derivative equal to zero gives

$$\frac{T}{\alpha} = \sum_{i=1}^{T} \log(Y_i/c)$$

and the maximum likelihood estimate for the right tail index is

$$\hat{\alpha} = \frac{T}{\sum_{i=1}^{T} \log(Y_i/c)}.$$

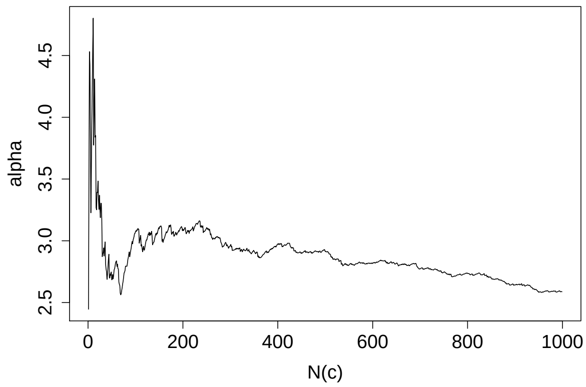

Figure 2: *Hill's plot* Hill's plot for the daily returns of S&P 500 for the time period starting at 1950-01-03 and ending at 2009-08-10, which makes 14997 observations. We make the Hill's plot for the 1000 smallest returns.

Hill's estimator for the left tail index is

$$\hat{\alpha} = \frac{N(c)}{\sum\{\log(-Y_i/c) : -Y_i \ge c\}},$$

where

$$N(c) = \#\{Y_i : -Y_i \ge c\}.$$

Hill's plot is the plot of points $(N(c), \hat{\alpha})$ for different values of $N(c)$, where $N(c)$ is the number of extreme returns used to estimate α. Figure 2 shows Hill's plot for the 1000 smallest returns of the daily returns of S&P 500 index for the time period starting at 1950-01-03 and ending at 2009-08-10, which makes 14997 observations. The plot starts to stabilize when the estimator is calculated from 300 observations and the value of the estimator is about 3. The value of the estimator decreases monotonicallly but stays above 2.5 when we use less than 1000 observations to calculate the estimator.

2.2 Value at Risk

We have defined value at risk in as the negative quantile of a random variable $Y \in \mathbf{R}$:

$$\mathrm{VaR}_p(Y) = -Q_p(Y),$$

where

$$Q_p(Y) = \inf\{y : P(Y \le y) \ge p\},$$

where $0 < p < 1$. The random variable Y is a return of an asset over a given time period. Thus the VaR has two parameters: the horizon over which the return is calculated (daily, weekly, 20 day horizon) and the confidence level p. The interpration of VaR in risk analysis is the following: we have a probability less than p of losing more than $\mathrm{VaR}_p(Y) \cdot P$ Euros at the end of the time horizon, where P is the number of invested Euros.

Let us consider estimation of VaR when we have a sample $Y_1, \ldots, Y_T$ of random variables equally distributed as Y. Note that it is an important question whether we can estimate VaR using a sample which has a more dense sampling frequency than the horizon of the return Y. For example, if Y is a 20 day return we could consider estimating the VaR using daily returns. However, we consider in the following the case where the sampling frequency is the same as the return period of Y.

1. *Nonparametric Estimate* We have defined a nonparametric estimate of value at risk as
$$\widehat{\mathrm{VaR}}_p(Y) = -Y_{([pT])}, \tag{6}$$
where we denote with $Y_{(1)}, \ldots, Y_{(T)}$ the order statistics (the sample values $Y_1, \ldots, Y_T$ in increasing order) and $[x]$ is the largest integer $\leq x$.

2. *Parametric Estimate* When $Y \sim N(\mu, \sigma^2)$, then we can estimate the VaR by
$$\widehat{\mathrm{VaR}}_p(Y) = -(\hat{\mu} + \Phi^{-1}(p)\,\hat{\sigma}),$$
where $\Phi : \mathbf{R} \to [0, 1]$ is the distribution function of standard Gaussian distribution, and $\hat{\mu}$ and $\hat{\sigma}$ are estimates of μ and σ, which can taken to be the arithmetic mean and the square root of the sample variance:
$$\hat{\mu} = \frac{1}{T}\sum_{i=1}^{T} Y_i, \qquad \hat{\sigma} = \left(\frac{1}{T}\sum_{i=1}^{T}(Y_i - \hat{\mu})^2\right)^{1/2}.$$

3. *Semiparametric Estimate* We have defined in (5) that the distribution of Y has Pareto left tail if
$$P(Y \leq -y) = L(y)\, y^{-\alpha},$$
for $y > 0$, where L is slowly varying and $\alpha > 0$ is the tail index. Let $0 < p_1 < p_2 < 1$ be close to zero. Let
$$y_1 = -Q_{p_1}(Y) = \mathrm{VaR}_{p_1}(Y), \qquad y_2 = -Q_{p_2}(Y) = \mathrm{VaR}_{p_2}(Y).$$

Now $y_1 > y_2$ are large. Assume that the distribution function of Y is continuous so that $P(Y \le -y_1) = p_1$ and $P(Y \le -y_2) = p_2$. Now

$$\frac{p_1}{p_2} = \frac{P(Y \le -y_1)}{P(Y \le -y_2)} = \frac{L(y_1)}{L(y_2)} \frac{y_1^{-\alpha}}{y_2^{-\alpha}} \approx \frac{y_1^{-\alpha}}{y_2^{-\alpha}}$$

which gives

$$\frac{\mathrm{VaR}_{p_1}(Y)}{\mathrm{VaR}_{p_2}(Y)} = \frac{y_1}{y_2} \approx \left(\frac{p_2}{p_1}\right)^{1/\alpha}$$

and thus

$$\mathrm{VaR}_{p_1}(Y) \approx \mathrm{VaR}_{p_2}(Y) \left(\frac{p_2}{p_1}\right)^{1/\alpha} .$$

Since $p_2 > p_1$ we can get a more accurate nonparamteric estimate for the quantile of confidence p_2 than for the quantile of p_1. Thus if we have an estimate $\hat{\alpha}$ for the tail index α and construct a nonparametric estimate $\widehat{\mathrm{VaR}}_{p_2}(Y)$ usinq (6), we get an estimate for the VaR at confidence p_1:

$$\widehat{\mathrm{VaR}}_{p_1}(Y) = \widehat{\mathrm{VaR}}_{p_2}(Y) \left(\frac{p_2}{p_1}\right)^{1/\hat{\alpha}} .$$

This estimate is called semiparametric, since we have used a semiparametric model for the tail.

3 Examination

Possible questions in the examination:

9) Define a distribution with Pareto right and left tails.

10) Define Hill's estimator of the tail index.

References

Bachelier, L. (1900), *Théorie de la Spéculation*, Gauthier-Villars.