Lecture 8

Jussi Klemelä

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1 Option Pricing: Single Period Hedging

We shall consider first the simple case of hedging an European option at a single time point (single period hedging).

- The underlying security has value S_0 at the beginning of the period and value S_1 at the expiration of the option. The price S_0 is a fixed number and S_1 is a random variable.
- At time zero the option seller sells the option at price H_0. The value of the European option at the expiration is denoted by H_1. For example, in the case of a call option $H_1 = \max\{S_1 - K, 0\}$, where K is the strike price. The price H_0 is negotiated between the seller and the buyer, or the price is determined in an option exchange based on supply and demand.
- The time between the beginning of the period and the end of the period is denoted by Δt and is expressed in fractions of a year. The annual risk free rate is denoted by r.

1.1 Pricing by Statistical Arbitrage

Let us approximate random variable H_1 by constructing a random variable from tradable assets. We approximate H_1 by a linear function of S_1:

$$a + bS_1,$$

where $a, b \in \mathbf{R}$. The approximation is made in L_2. We solve the problem

$$\min_{a,b \in \mathbf{R}} E(a + bS_1 - H_1)^2.$$

We get the equations

$$\begin{cases} E(a + bS_1 - H_1) = 0, \\ E[(a + bS_1 - H_1)\, S_1] = 0. \end{cases}$$

We get the solutions

$$b = \frac{\mathrm{Cov}(S_1, H_1)}{\mathrm{Var}(S_1)}, \qquad a = EH_1 - bES_1. \tag{1}$$

The coefficient b gives the optimal hedging coefficient. The price H_0 for the option is the (discounted) value of the random variable $a + bS_1$ at time 0:

$$\begin{aligned} H_0 &= e^{-r\Delta t} a + bS_0 \\ &= e^{-r\Delta t} EH_1 + \frac{\mathrm{Cov}(S_1, H_1)}{\mathrm{Var}(S_1)} \left(S_0 - e^{-r\Delta t} ES_1\right). \end{aligned}$$

1.2 Pricing by Variance Optimal Hedging

The principle of variance hedging leads to the Bouchaud-Sornette pricing. We discuss the case from the point of view of the option seller. The wealth balance of the option seller shall be determined by the following three transactions.

1. The option seller receives the option premium H_0 at time 0. The premium is invested in a risk free bond, so that the option seller receives $e^{r\Delta t} H_0$ at time T.

2. The option seller has to pay the amount of H_1 dollars at time 1.

3. The option seller can hedge the option at time 0 by borrowing with the risk free rate and investing that money in the underlying stock. We denote with ξS_0, $\xi \in \mathbf{R}$, the amount which is borrowed and invested in the stock. This transaction results in the payment

   $$S_0 \xi \left(\frac{S_1}{S_0} - e^{r\Delta t} \right). \tag{2}$$

   at time 1.

   Note that if we have initial wealth S_0 and invest the proportion $\xi \in [0, 1]$ to the stock and the proportion $1 - \xi$ to the bond, then wealth at time 1 is

   $$S_0 \left(\xi \cdot \frac{S_1}{S_0} + (1 - \xi) \cdot e^{r\Delta t} \right).$$

Hovever, since the amount S_0 is borrowed at time 0, the amount $e^{r\Delta t}S_0$ have to be paid byck at time 1, and at time 1 the wealth is equal to (2).

Collecting the three transactions together, we get the wealth at time 1:

$$W_1 = e^{r\Delta t}H_0 + \xi S_0 \left(\frac{S_1}{S_0} - e^{r\Delta t}\right) - H_1. \tag{3}$$

The hedging parameter is chosen so that the variance of the final wealth is minimized. Thus we define the optimal hedging parameter as

$$\xi^* = \mathrm{argmin}_{\xi \in \mathbf{R}} \mathrm{Var}(W_1).$$

The variance of the final wealth can be written as

$$\mathrm{Var}(W_1) = \mathrm{Var}(\xi X_1 - H_1), \tag{4}$$

where we denote

$$X_1 = S_0 \left(\frac{S_1}{S_0} - e^{r\Delta t}\right). \tag{5}$$

The factor ξ minimizing (4) is

$$\xi^* = \frac{\mathrm{Cov}(X_1, H_1)}{\mathrm{Var}(X_1)} = \frac{\mathrm{Cov}(S_1, H_1)}{\mathrm{Var}(S_1)}. \tag{6}$$

Note that $\xi^* \in [0, 1]$, because it holds always that $0 \le H_1 \le S_1$. [1] The price H_0 is chosen so that $EW_1 = 0$, that is,

$$H_0 = e^{-r\Delta t} E\left[H_1 - \xi^* S_0 \left(\frac{S_1}{S_0} - e^{r\Delta t}\right)\right]. \tag{7}$$

1.3 Empirical Analysis of Single Period Hedging

1.3.1 Linear Approximation of Option Payoff

Figure 1 shows the best linear approximations to the option payoffs H_1. We show the scatter plots of (S_1, H_1) and $(S_1, a + bS_1)$, where a and b are defined in (1), when we have simulated 1000 points from the distribution of S_1. The distribution of S_1 is the logarithmic Gaussian distribution with three volatility parameters: in panel (a) the annual volatility is 0.1, in panel (b) it is 0.2, and in panel (c) it is 0.3.

[1] By the Cauchy-Schwartz inequality $\mathrm{Cov}(S_1, H_1) \le \sqrt{\mathrm{Var}(S_1)\mathrm{Var}(H_1)}$, and since $0 \le H_1 \le S_1$ we have $\mathrm{Var}(H_1) \le \mathrm{Var}(S_1)$.

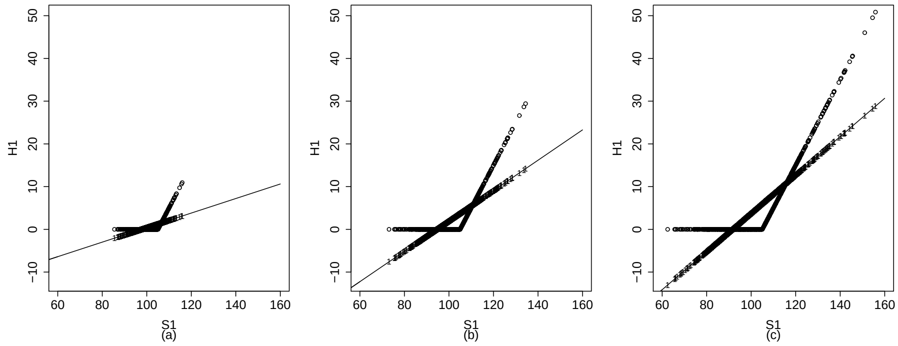

Figure 1: *Linear Approximation of* H_1 Scatter plots of $(S_1, (S_1 - K)_+)$ and $(S_1, a + bS_1)$ when we have simulated a sample of size 1000 from the distribution of S_1, which is the logarithmic Gaussian distribution with the annual volatility of (a) 0.1, (b) 0.2, and (c) 0.3. The linear fits have (a) $b = 0.17$, (b) $b = 0.36$, and (c) $b = 0.45$. We have drawn also the lines $y = a + bx$ to the plots.

The other parameters of the stock price process and the option are: the drift is taken to be zero ($m = 0$), the initial stock price is $S_0 = 100$, the option is the call option with strike price $K = 105$, so that $H_1 = \max\{0, S_1 - K\}$, the time to expiration is $\Delta t = 0.25$ in fractions of year, and we take the number of trading days in year to be 250, so that the number of trading days to expiration is 62, and the annual risk free interest rate is $r = 0.05$.

In Panel (a) $b = 0.17$, in panel (b) $b = 0.36$, and in panel (c) $b = 0.45$. We have drawn also the lines $y = a + bx$ to the plots. These values of coeffcient b are almost equal to the parameter delta of the Black-Scholes pricing model: we have $\delta = 0.24$, $\delta = 0.38$, and $\delta = 0.43$ for the three cases.

1.3.2 Optimal Hedging Coefficient

We shall plot the optimal hedging coefficient as the function of moneyness index S_0/K, where S_0 is the stock price at the time of heding and K is the strike price of the European call option. The optimal hedging coefficient is defined in (6). The risk free interest rate is taken to be 0 in the following examples. The moneyness index varies in the range $0.6-1.5$. In general, when the moneyness index is smaller than 1 (the option is out of the money, $S_0 > K$), then the hedging coefficient is smaller than 0.5. When the moneyness is larger than 1 (the option is in the money, $S_0 > K$), then the hedging coefficient is larger than 0.5.

1. Figure 2 shows the optimal hedging coefficients for the case when the price has a logarithmic Gaussian distribution.

   Panel (a) shows the case where the time to expiration is 0.25 in fractions of a year, and the annualized volatility takes values $0.05, 0.15, 0.25, 0.35,$ and 0.45. We plot the hedging parameter as the function of moneyness for each volatility parameter, so that there are 5 curves in the panel. Panel (a) shows that when the volatility is small, then the function is almost a step function and the value of the hedging coefficient jumps from 0 to 1 at point 1. When the volatility is large, then the values around 0.5 are optimal for a large range of moneyness index around 1.

   Panel (b) shows the case where the annualized volatility is 0.2 and the time to expiration takes values 0.2, 0.4, 0.6, 0.8, and 1. We plot the hedging parameter as the function of moneyness for each maturity parameter. Panel (b) shows that when the time to expiration is small, then the hedging coefficient takes for almost all values of the moneyness only values 0 and 1. When the time to expiration is large, then the values around 0.5 are optimal for a large range of moneyness indexes around 1.

2. Figure 3 shows the optimal hedging coefficients when the price has a logarithmic Student distribution, so that

   $$\log S_1 - \log S_0 \sim \mu + \sigma\epsilon,$$

   where ϵ has the Student distribution and $\sigma = 0.2 \times \sqrt{250}$ (the annualized volatility is 0.2).

   In panel (a) the degrees of freedom are 3, 4, and 6. For comparison, we show also the case of the logarithmic Gaussian process with the same volatility parameter. Panel (a) shows that in the Gaussian case the curve is steepest and in the student case a wider range of the hedging coefficient around 0.5 has to be considered. The curve is flattest when the degrees of freedom is 3.

   In panel (b) the degrees of freedom is 4 and the annualized volatility takes the values -0.1, 0, 0.1, and 0.2. Panel (b) shows that in the Student case the drift affects the hedging parameter, although in the Gaussian case it is known that the drift does not affect hedging. When the drift is positive, then the hedging coefficient has to be taken larger and when the drift is negative, then the hedging parameter has to be taken smaller.

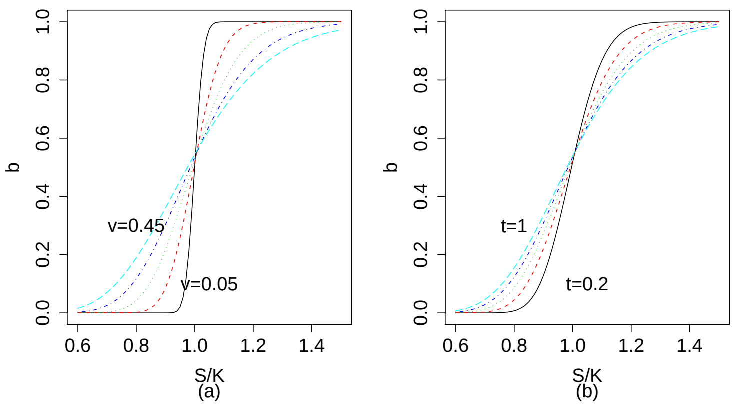

Figure 2: *Hedging Coefficient; Gaussian Case* The hedging coefficient is shown as the function of moneyness. (a) The values of annualized volatility are 0.05, 0.15, 0.25, 0.35, and 0.45, when the time to expiration is 0.25 in fractions of year. (b) The values of the maturity parameter are 0.2, 0.4, 0.6, 0.8, and 1, and the volatility parameter is 0.2. The stock price has a logarithmic Gaussian distribution.

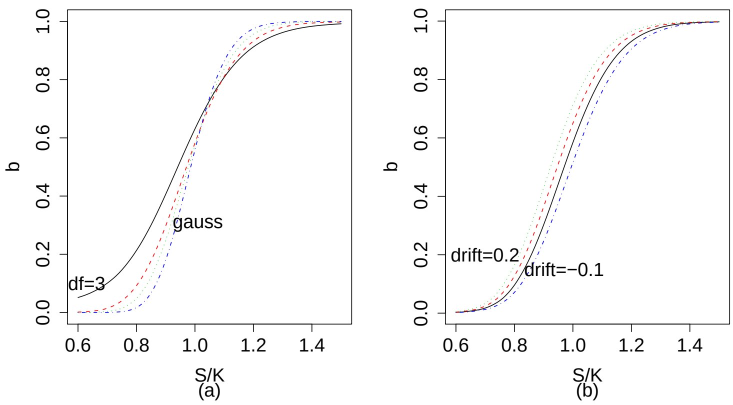

Figure 3: *Hedging Coefficient; Student Case* The hedging coefficient is shown as the function of moneyness. (a) The degrees of freedom are 3, 4, and 6; (b) the annualized drift takes values -0.2, 0.1, 0, 0.1 and the degrees of freedom is 4. The stock price has a logarithmic Student distribution. In panel (a) also the Gaussian case is is shown. In both panels the annualized volatility is 0.2 and the time to maturity is 0.25 in fractions of year.

1.3.3 Wealth Distribution

We study the probability distribution of the wealth of the writer of the option at the expiration. We choose the parameters of the underlying stock price distribution and the option to be the same as in Figure 1, except that now we consider non-zero expected return of the stock price. Figure 4 considers the case of the annualized return equal to $m = 0.1$ and Figure 5 considers the case of the annualized return equal to $m = -0.1$. (The annualized return is equal to 250 times the daily return.)

Figures 4-5 show the wealth distributions when we have simulated 1000 realizations of the stock prices. Unlike in (3) we ignore the price of the option in the presentation of the distribution, because the price is a constant which can vary in practise due to the market demand. The number of bins of the histograms is in all cases 25.

1. Figure 4 considers the case when the annualized return is equal to 0.1. Panel (a) shows a histogram of the sample from the distribution of $-H_1$ and panel (b) shows a histogram of the sample from the distribution of ξX_1, where X_1 is defined in (5) and ξ is estimated using (6), which gives $\xi = 0.45$. Panel (c) shows a histogram of the realizations of ξX_1.

   Panel (a) shows that $-H_1$ takes value 0 with a large probability, and large negative values with a small probability. The sample mean of the realizations of $-H_1$ is -3.01. The sample mean of the realizations of ξX_1 in panel (b) is 0.75. The price estimated as a sample mean from (7) is 2.32 and the Black-Scholes price is 2.48.

2. Figure 5 considers the case when the annualized return is equal to -0.1. Panel (a) shows a histogram of the sample of realizations $-H_1$ and panel (b) shows a histogram of the realizations of ξX_1, where ξ is estimated using (6), which gives $\xi = 0.27$. Panel (c) shows a histogram of the realizations of $\xi X_1 - H_1$.

   Panel (a) shows that $-H_1$ does not take as large negative values as in the case of positive stock returns. The sample mean of the realizations of $-H_1$ is -1.38. However, panel (b) shows that the effect of trading is now negative on the expected wealth: the sample mean of the realizations of ξX_1 in panel b) is -0.88. The price estimated as a sample mean from (7) is 2.23 and the Black-Scholes price is again 2.48.

The examples show that the fair price of the option does not depend on the expected return of the stock. In the case of a negative expected return the hedging has a negative effect on the expected wealth of the hedger, but

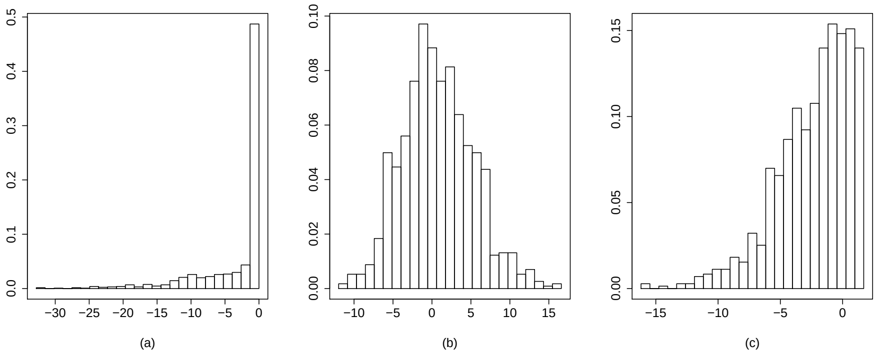

Figure 4: *Hedging with Positive Returns* Histograms of simulated data of size 1000 from the distribution of (a) $-H_1$, (b) ξX_1, and (c) $\xi X_1 - H_1$, when the stock has the logarithmic Gaussian distribution with the annualized drift 0.1 and annualized volatility 0.2.

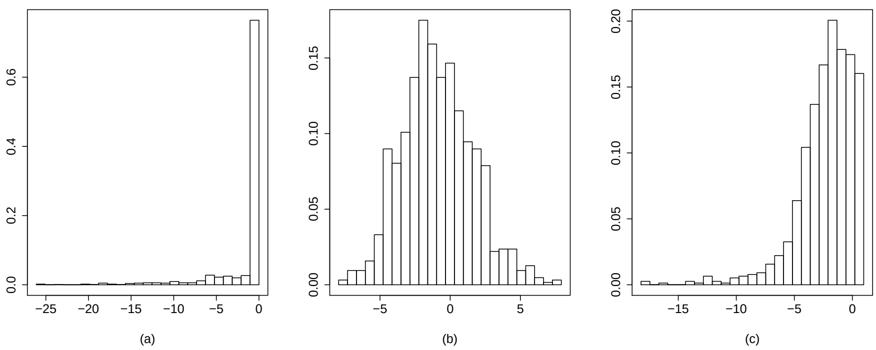

Figure 5: *Hedging with negative returns* Histograms of simulated data of size 1000 from the distribution of (a) $-H_1$, (b) ξX_1, and (c) $\xi X_1 - H_1$, when the stock has the logarithmic Gaussian distribution with the annualized drift -0.1 and annualized volatility 0.2.

in both cases hedging reduces the variance of the wealth, as it is designed to do.

2 Examination

Possible questions in the examination:

12) Discuss option pricing in the single period model: Find a and b that solve the problem
$$\min_{a,b\in\mathbf{R}} E(a + bS_1 - H_1)^2,$$
where H_1 is the option price at the expiration and S_1 is the stock price at the expiration. Give the corresponding price H_0 of the option at time 0.