Lecture 9

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1 Two Period Hedging

We consider the case of hedging an European option at two time points (two period hedging).

- The underlying security takes values S_0, S_1, and S_2 at consecutive time points $0, 1, 2$. The option is written at time 0 and it expires at time 2. The price S_0 is a fixed number and S_1 and S_2 are random variables.
- At time zero the writer of the option sells the option at price H_0. The value of the European option at the expiration is denoted by H_2. For example, in the case of a call option $H_2 = \max\{S_2 - K, 0\}$, where K is the strike price.
- The time between the time points 0, 1, and 2 is denoted by Δt and is expressed in fractions of a year. The annual risk free rate is denoted by r.

The option price H_2 is determined at time point 2 and we can first approximate random variable H_2 with $a_2 + b_2 S_2$, where $a_2, b_2 \in \mathbf{R}$. We solve the problem

$$\min_{a_2, b_2 \in \mathbf{R}} E_1 (a_2 + b_2 S_2 - H_2)^2,$$

where the expectation is taken conditional on the information available at time 1. Similarily as in the single period hedging we get

$$b_2 = \frac{\mathrm{Cov}_1(S_2, H_2)}{\mathrm{Var}_1(S_2)}, \qquad a_2 = E_1 H_2 - b_2 E_1 S_2.$$

Note that now the expectations, variances, and covariances are conditional on the information available at time 1. Denote by H_1 the discounted value to time point 1 of the random variable $a_2 + b_2 S_2$:

$$\begin{aligned} H_1 &= e^{-r\Delta t} a_2 + b_2 S_1 \\ &= e^{-r\Delta t} E_1 H_2 + \frac{\mathrm{Cov}_1(S_2, H_2)}{\mathrm{Var}_1(S_2)} \left(S_1 - e^{-r\Delta t} E_1 S_2 \right). \end{aligned}$$

The next step is to approximate H_1 by random variable $a_1 + b_1 S_1$, where $a_1, b_1 \in \mathbf{R}$. We get

$$b_1 = \frac{\mathrm{Cov}_0(S_1, H_1)}{\mathrm{Var}_0(S_1)}, \qquad a_1 = E_0 H_1 - b_1 E_0 S_1.$$

The price H_0 is

$$\begin{aligned} H_0 &= e^{-r\Delta t} a_1 + b_1 S_0 \\ &= e^{-r\Delta t} E_0 H_1 + \frac{\mathrm{Cov}_0(S_1, H_1)}{\mathrm{Var}_0(S_1)} \left(S_0 - e^{-r\Delta t} E_0 S_1 \right). \end{aligned}$$

2 Wealth Process

Let $S_t = (S_t^1, \ldots, S_t^d)$ be the vector of tradable assets. We can write the wealth process both in a multiplicative way and in an additive way. We have previously used the multiplicative wealth process. This approach is valid for the cases where we have a positive initial wealth $W_0 > 0$. Now we write the wealth process in an additive way. This way of writing the wealth process does not presuppose a positive initial wealth.

2.1 General Wealth Process

Let ξ_{t+1} be the vector whose elements give the numbers of stocks invested at time t. We require that the strategy is self-financing, which means that the vector ξ_{t+1} satisfies at time t

$$W_t = \xi_{t+1}^T S_t. \tag{1}$$

The self-financing property means that we can invest only the available wealth W_t. Note, however, that if we allow negative values for ξ_{t+1}^i, then borrowing and short selling is allowed. Now

$$W_t = W_0 + \sum_{i=1}^{t} (W_i - W_{i-1}) = W_0 + \sum_{i=1}^{t} \xi_i^T (S_i - S_{i-1}),$$

where we used the facts $W_{i-1} = \xi_i^T S_{i-1}$ and $W_i = \xi_i^T S_i$.

2.2 Wealth Process with Risk Free Rate

The self-financing property (1) of the weights sets a linear constraint for the portfolio vector ξ_{t+1}, which reduces its dimension by one. Next we assume that we have $d+1$ tradable assets and one of the assets is the risk free investment. We write the wealth balance using only the d dimensional portfolio vector. We shall denote by ξ_{t+1} the d dimensional column vector whose ith element is equal to the number of stock i bought at time t. We shall denote with a_{t+1} the number of bonds bought at time t. The bond is defined by

$$B_{t+1} = (1+r)B_t.$$

The self financing property requires that

$$W_t = a_{t+1}B_t + \xi_{t+1}^T S_t,$$

that is, we invest exactly the wealth W_t available at time t. Now,

$$\begin{aligned} W_{t+1} &= a_{t+1}B_{t+1} + \xi_{t+1}^T S_{t+1} \\ &= a_{t+1}(1+r)B_t + \xi_{t+1}^T S_{t+1} \\ &= (1+r)(a_{t+1}B_t + \xi_{t+1}^T S_t) + \xi_{t+1}^T [S_{t+1} - (1+r)S_t] \\ &= (1+r)W_t + \xi_{t+1}^T [S_{t+1} - (1+r)S_t]. \end{aligned} \tag{2}$$

We can write this equation in the one dimensional case $d = 1$ also as

$$W_{t+1} = (1+r)W_t + \xi_{t+1}S_t \left(\frac{S_{t+1}}{S_t} - (1+r) \right),$$

which can be interpreted as an transaction where W_t is invested in the bond, the amount $\xi_{t+1}S_t$ is borrowed and invested in the stock. From (2) we get inductively

$$W_t = W_0(1+r)^t + \sum_{k=0}^{t-1} \xi_{k+1}^T (1+r)^{t-k-1}[S_{k+1} - (1+r)S_k]. \tag{3}$$

We can write the Equation (3) in the following equivalent or related ways.

- Denote $X_k = (1+r)^{-k}S_k$. Then,

$$W_t = (1+r)^t \left(W_0 + \sum_{k=0}^{t-1} \xi_{k+1} \cdot (X_{k+1} - X_k) \right). \tag{4}$$

- We may write the wealth balance using the continuous time discounting as

$$W_t = W_0 e^{rt} + \sum_{k=0}^{t-1} \xi_{k+1} \cdot e^{r(t-k-1)} (S_{k+1} - e^r S_k).$$

- Let us use the continuous time discounting and make the time step explicit. We divide the time interval $[0,t]$ to N steps of length τ: $t_k = \tau k$, $k = 0, \ldots, N$, $N = t/\tau$, $\tau > 0$. We assume that the trading may be done at the time points t_k. Then

$$W_t = W_0 e^{rt} + \sum_{k=0}^{N-1} \xi_{k+1} \cdot e^{r(t-t_k-\tau)} (S_{t_{k+1}} - e^{r\tau} S_{t_k}).$$

3 Variance Optimal Hedging

Here we follow Bouchaud and Potters (2003). When there exists a trading strategy which replicates the option pay-off, then the fair price of the contract is such that there are no arbitrage opportunities. If there does not exist a trading strategy replicating the option pay-off, then the price is such that the average profit, including the cost of the trading strategy, is zero.

The wealth balance W_T, obtained after trading with a bond and stocks, was written in (3) and (4). Now we assume that there is only one stock ($d = 1$) and in addition we write an European option on this stock, expiring at time T. The price of the option at the time of writing is denoted by H_0 and the value of the option at the expiry is denoted by H_T. For example, when the option is the European call option with the strike price K, then

$$H_T = \max\{0, S_T - K\}.$$

Then the wealth of the writer of the option at time T is

$$\mathcal{W}_T = W_T + (1+r)^T H_0 - H_T.$$

The wealth consists of three terms, the wealth obtained by trading (hedging), the wealth obtained from the received option premium, and the payment of the option value at the expiry. We choose the trading strategy ξ^* that minimzes the variance of the wealth at the expiry:

$$\xi^* = \mathrm{argmin}_\xi \mathrm{Var}\left(\mathcal{W}_T\right).$$

The optimal trading strategy does not depend on H_0, which is a constant. We shall later consider also other risk functionals. Let W_T^* be the wealth obtained by using an optimal trading strategy ξ^* and taking $W_0 = 0$:

$$W_T^* = (1+r)^T \sum_{k=0}^{T-1} \xi_{k+1}^* \cdot (X_{k+1} - X_k).$$

The fair option price H_0 is determined from the equation

$$E\mathcal{W}_T^* = EW_T^* + (1+r)^T H_0 - EH_T = 0. \tag{5}$$

3.1 Variance Optimal Price

We get the option price from (5) as

$$\begin{aligned} H_0 &= (1+r)^{-T} \left(EH_T - EW_T^* \right). \\ &= (1+r)^{-T} EH_T - \sum_{k=0}^{T-1} E\left[\xi_{k+1}^* \cdot (X_{k+1} - X_k) \right]. \end{aligned} \tag{6}$$

Note that we can write

$$E\left[\xi_{k+1}^* \cdot (X_{k+1} - X_k)\right] = E\left[\xi_{k+1}^* \cdot E(X_{k+1} - X_k \,|\, \mathcal{F}_k)\right].$$

Thus we can simplify the price under some assumptions.

1. If the incerements $X_{k+1} - X_k$ are identically distributed with

   $$\mu = E(X_{k+1} - X_k)$$

   then

   $$H_0 = (1+r)^{-T} EH_T - \mu \sum_{k=0}^{T-1} E\left[\xi_{k+1}^*\right].$$

2. If $X_{k+1} - X_k$ is a martingale, which means that

   $$E(X_{k+1} - X_k \,|\, \mathcal{F}_k) = 0, \tag{7}$$

   then we get that

   $$E\left[\xi_{k+1}^* \cdot (X_{k+1} - X_k)\right] = E\left[\xi_{k+1}^* \cdot E(X_{k+1} - X_k \,|\, \mathcal{F}_k)\right] = 0$$

   and the fair price is

   $$H_0 = (1+r)^{-T} EH_T, \tag{8}$$

   which does not depend on the trading strategy.

3.2 Optimal Hedging

We want to find the trading strategy ξ^* which minimizes

$$\mathrm{Var}(\mathcal{W}_T) = \mathrm{Var}(W_T - H_T)^2.$$

We assume that $X_k - X_{k-1}$ is a martingale as in (7), which implies that the increments are uncorrelated: for $l \le k-1$

$$\begin{aligned} & E\left[\xi_{l+1}(X_{l+1} - X_l)\, \xi_{k+1}(X_{k+1} - X_k)\right] \\ = \;& E\left[\xi_{l+1}(X_{l+1} - X_l)\; \xi_{k+1} E(X_{k+1} - X_k \,|\, \mathcal{F}_k)\right] = 0. \end{aligned} \tag{9}$$

3.2.1 Optimal Trading Strategy

We have that

$$\mathrm{Var}(W_T - H_T) = \mathcal{R}_1 + \mathcal{R}_2,$$

where

$$\mathcal{R}_1 = \mathrm{Var}(H_T)$$

and

$$\mathcal{R}_2 = \mathrm{Var}(W_T) - 2\mathrm{Cov}(W_T, H_T).$$

Term $\mathcal{R}_1$ is the bare risk which the writer of the option has to take independently of trading. The optimal strategy minimizes the term $\mathcal{R}_2$ with respect to ξ_t. From (9) we get

$$\mathcal{R}_2 = \sum_{k=0}^{T-1} E\left[\xi_{k+1}(X_{k+1} - X_k)\right]^2 - 2\sum_{k=0}^{T-1} E\left[\tilde{H}_T\, \xi_{k+1}(X_{k+1} - X_k)\right], \tag{10}$$

where

$$\tilde{H}_T = (1+r)^{-T} H_T.$$

For example, when $H_T = (S_T - K)_+$, then $\tilde{H}_T = (X_T - \tilde{K})_+$, where $\tilde{K} = (1+r)^{-T}K$. In addition to assumption (9) we make also the following assumption:

$$\xi_{k+1} \text{ is a function of } X_k \text{ only (and not a function of } X_{k-1}, \ldots, X_0). \tag{11}$$

Then we may write

$$\begin{aligned} E\left[\xi_{k+1}(X_{k+1} - X_k)\right]^2 &= \int \xi_{k+1}^2(x)(y-x)^2 f_{X_{k+1},X_k}(y,x)\,dx \\ &= \int \xi_{k+1}^2(x) D_{k+1}(x) f_k(x \,|\, X_0 = x_0)\,dx \end{aligned}$$

where

$$D_{k+1}(x) = \int (y-x)^2 f_{k+1}(y \,|\, X_k = x)\, dy,$$

since we have written the joint density of (X_{k+1}, X_k) in terms of the conditional densities:

$$f_{X_{k+1},X_k}(y,x) = f_{k+1}(y \,|\, X_k = x)\, f_k(x \,|\, X_0 = x_0),$$

where $x_0 = S_0$. Write

$$\begin{aligned} & E\left[\tilde{H}_T(X_T)\, \xi_{k+1}(X_k)(X_{k+1} - X_k)\right] \\ & = \int \xi_{k+1}(x) f_k(x \,|\, X_0 = x_0) \int (y-x) f_{k+1}(y \,|\, X_k = x) \\ & \quad \times \int \tilde{H}_T(z) f_T(z \,|\, X_{k+1} = y)\, dz dy dx, \end{aligned}$$

where we have used the decomposition

$$f_{X_T,X_{k+1},X_k}(z,y,x) = f_T(z \,|\, X_{k+1} = y) f_{k+1}(y \,|\, X_k = x) f_k(x \,|\, X_0 = x_0).$$

Derivating (10) inside the summation and inside the integral with respect to ξ_{k+1}, and setting the derivative to zero leads to

$$\begin{aligned} & \xi_{k+1}(x) f_k(x \,|\, X_0 = x_0) D_{k+1}(x) \\ & - \; f_k(x \,|\, X_0 = x_0) \int (y-x) f_{k+1}(y \,|\, X_k = x) \\ & \quad \times \int \tilde{H}_T(z) f_T(z \,|\, X_{k+1} = y)\, dz dy = 0. \end{aligned}$$

Solving this gives the optimal trading strategy ξ^*_{k+1} as

$$\begin{aligned} & \xi^*_{k+1}(x) \\ & = \frac{1}{D_{k+1}(x)} \int_{-\infty}^{\infty} (y-x) f_{k+1}(y \,|\, X_k = x) \int_{-\infty}^{\infty} \tilde{H}_T(z) f_T(z \,|\, X_{k+1} = y)\, dz dy \\ & = \frac{1}{D_{k+1}(x)} \; E\left[\tilde{H}_T(X_T)(X_{k+1} - X_k) \,\Big|\, X_k = x\right], \end{aligned}$$

where

$$D_{k+1}(x) = \; E\left[(X_{k+1} - X_k)^2 \,\big|\, X_k = x\right].$$

3.2.2 Identically Distributed Increments

We assume that the increments $X_{k+1} - X_k$, $k = 0, \ldots, T-1$, are i.i.d. and that the density of an increment is

$$\phi_\sigma = \frac{1}{\sigma}\,\phi\left(\frac{\cdot}{\sigma}\right) \tag{12}$$

for some $\sigma > 0$. Now we have

$$\xi^*_{k+1}(x) = \frac{1}{(T-k)\sigma^2}\int_{-\infty}^{\infty}(z-x)\tilde{H}(z)f_T(z\,|\,X_k = x)\,dz. \tag{13}$$

3.2.3 Delta Hedging

We have shown in (8) that under the martingale assumption the price of the option is

$$H_t = (1+r)^{-(T-t)}E_t H_T = \int_{-\infty}^{\infty}\tilde{H}_T(z)f_T(z\,|\,X_t = x)\,dz,$$

where $x = (1+r)^{-(T-t)}S_t$ and S_t is the stock price at time t. We make the further assumption that the increments $X_{k+1} - X_k$ are i.i.d. with the density given in (12) and ϕ is the standard Gaussian density given by $\phi(z) = (2\pi)^{-1/2}\exp\{-z^2/2\}$. Then $X_T = \sum_{k=t}^{T-1}(X_{k+1} - X_k) + X_t$ and

$$f_T(z\,|\,X_t = x) = \frac{1}{\sigma\sqrt{T-t}}\,\phi\left(\frac{z-x}{\sigma\sqrt{T-t}}\right).$$

We have

$$\begin{aligned}\frac{\partial}{\partial x}H_t(x) &= \int_{-\infty}^{\infty}\tilde{H}_T(z)\,\frac{\partial}{\partial x}f_T(z\,|\,X_t = x)\,dz\\ &= \frac{1}{\sigma^2(T-t)}\int_{-\infty}^{\infty}\tilde{H}_T(z)(z-x)f_T(z\,|\,X_t = x)\,dz,\end{aligned}$$

since

$$\begin{aligned}\frac{\partial}{\partial x}f_T(z\,|\,X_t = x) &= -\frac{1}{\sigma^2(T-t)}\,\phi'\left(\frac{z-x}{\sigma\sqrt{T-t}}\right)\\ &= \frac{1}{\sigma^2(T-t)}\,\frac{z-x}{\sigma\sqrt{T-t}}\,\phi\left(\frac{z-x}{\sigma\sqrt{T-t}}\right)\\ &= \frac{1}{\sigma^2(T-t)}\,(z-x)\,f_T(z\,|\,X_t = x),\end{aligned}$$

where we used $\phi'(x) = -x\phi(x)$. From (13) we get

$$\xi_{t+1}(x) = \frac{\partial}{\partial x}H_t(x).$$

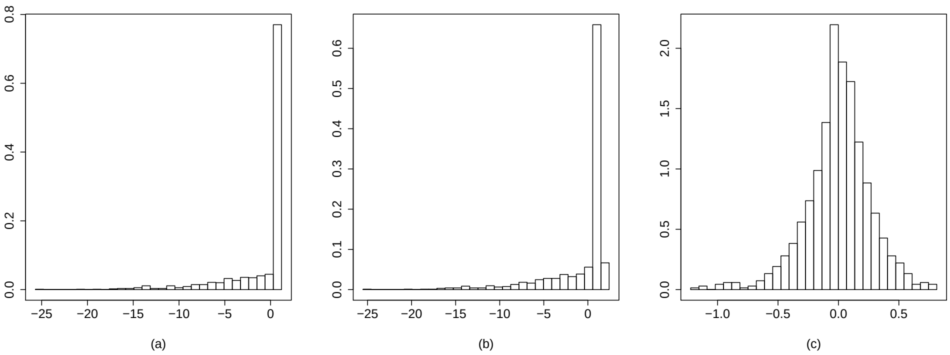

Figure 1: *Delta Hedging* A histogram from the realizations of (a) the wealth $e^{rT}H_0 - (S_T - K)_+$ of the writer of the option when the position is not hedged, (b) the final value W_T of the hedging process, and (c) the wealth $e^{rT}H_0 + W_T - (S_T - K)_+$ of the writer of the option when hedging is performed. We simulate 1000 trajectories from a log-normal process with the annualized drift 0.08.

4 Empirical Analysis of Hedging

4.1 Multiperiod Hedging

We illustrate hedging in the case of a call option with the payoff $(S_T - K)_+$ when the stock price follows a geometric Brownian motion.

When the drift is larger than the risk free rate, $E(S_T - K)_+$ is larger than $e^{rT}H_0$. The possibility of hedging makes the price smaller than $E(S_T - K)_+$. The expectation $E(S_T - K)_+$ increases when the drift increases but the possibility of hedging makes the price independent of the drift. When the drift is equal to the risk free rate, then $E(S_T - K)_+$ is close to $e^{rT}H_0$, but hedging reduces the risk of the writer of the option because it changes the wealth distribution of the writer of the option. When the drift is negative, then $E(S_T - K)_+$ is smaller than $e^{rT}H_0$, and hedging reduces the expected profit of the writer of the option. However, the hedging reduces also the risk of the writer of the option, and thus hedging is reasonable even in the case of negative drift.

1. Figure 1 illustrates the effect of delta hedging. We simulated 1000 repetitions from a log-normal process with annual drift of 0.08 and annual volatility of 0.15. The initial stock price is $S_0 = 100$, the strike price is $K = 105$, there are $T = 90$ days to maturity, and the risk free rate is $r = 0$.

Panel (a) shows a histogram of the wealth of the writer of a call option at the expiration when no hedging is done. The histogram is made from 1000 realizations of the random variable $e^{rT}H_0 - (S_T - K)_+$. The wealth of the writer of the call option at the expiration is equal to the option premium minus the pay-off of the option. The price H_0 is taken to be the Black Scholes price.

Panel (b) shows a histogram of the payout distribution of the delta hedging strategy, with the same 1000 trajectories that produced the histogram of panel (a). The delta hedging is actualized once a day (there is one trade at each day). The histogram is made from 1000 observations of the random variable W_T, where W_t, $0 \le t \le T$, is the wealth process with the Black Scholes hedging strategy ξ_t, $0 \le t \le T$.

Panel (c) shows a histogram of the wealth of the writer of the call option at the expiration when the option is hedged. The histogram is made from 1000 realizations of the random variable $W_T + e^{rT}H_0 - (S_T - K)+$.

The mean over the 1000 realization of the payout of the call is $E(S_T - K)_+ = 1.93$, the mean of the final wealth of delta hedging strategy is $EW_T = 0.73$. and the difference of these numbers is 1.194, which is very close to the Black-Scholes price 1.193.

2. Figure 2 shows the setting of Figure 1 when the annualized drift is 0.5, instead of 0.08. In this case the call option gives a profit to its owner with a large probability but this does not affect the price of the option.

   The mean over the 1000 realization of the payout of the call is $E(S_T - K)_+ = 9.05$, the mean of the final wealth of delta hedging strategy is $EW_T = 7.82$. and the difference of these numbers is 1.23, which is very close to the Black-Scholes price 1.193.

3. Figure 3 shows the setting of Figure 1 when the annualized drift is -0.08, instead of 0.08. In this case the call option gives a profit to the writer of the option with a large probability but this does not affect the price of the option.

   The mean over the 1000 realization of the payout of the call is $E(S_T - K)_+ = 0.79$, the mean of the final wealth of delta hedging strategy is $EW_T = -0.39$. and the difference of these numbers is 1.187, which is very close to the Black-Scholes price 1.193.

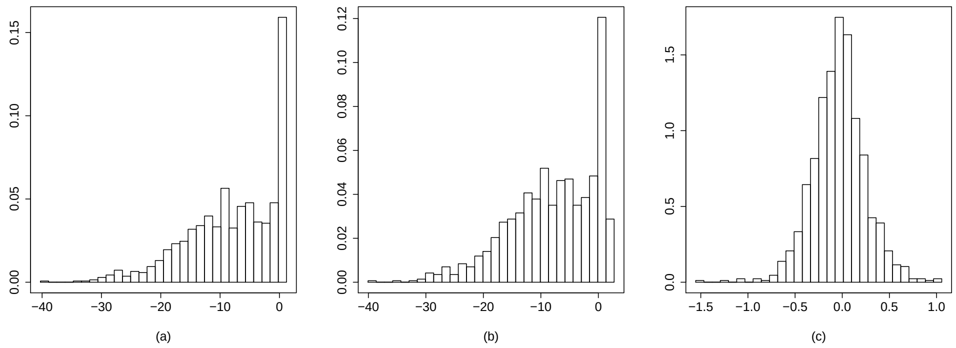

Figure 2: *Delta Hedging with Large Drift* The setting of Figure 1 when the annualized drift is 0.5, instead of 0.08. A histogram from the realizations of (a) the wealth $e^{rT}H_0 - (S_T - K)_+$ of the writer of the option when the position is not hedged, (b) the final value W_T of the hedging process, and (c) the wealth $e^{rT}H_0 + W_T - (S_T - K)_+$ of the writer of the option when hedging is performed. We simulate 1000 trajectories from a log-normal process.

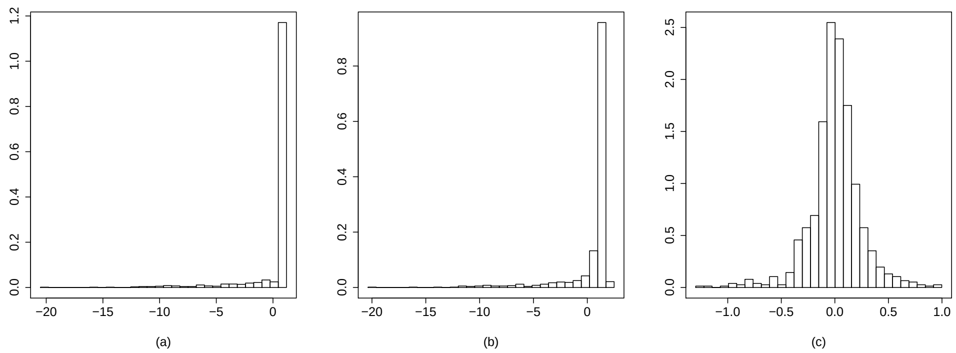

Figure 3: *Delta Hedging with Negative Drift* The setting of Figure 1 when the annualized drift is -0.08, instead of 0.08. A histogram from the realizations of (a) the wealth $e^{rT}H_0 - (S_T - K)_+$ of the writer of the option when the position is not hedged, (b) the final value W_T of the hedging process, and (c) the wealth $e^{rT}H_0 + W_T - (S_T - K)_+$ of the writer of the option when hedging is performed. We simulate 1000 trajectories from a log-normal process.

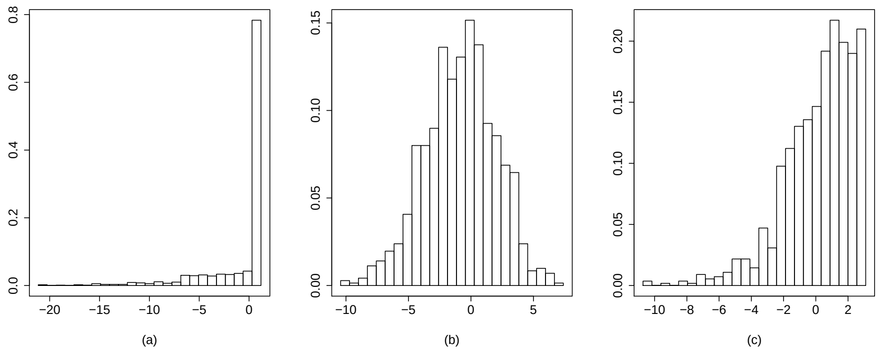

Figure 4: *Single Period Hedging* Histograms of simulated data of size 1000 from the distribution of (a) $e^{rT}H_0 - H_1$, (b) ξX_1, and (c) $e^{rT}H_0 + \xi X_1 - H_1$, when the stock has the logarithmic Gaussian distribution with the annualized drift 0.08 and annualized volatility 0.15.

4.2 Single Period Hedging

We study the probability distribution of the wealth of the writer of the option at the expiration. We choose the parameters of the underlying stock price distribution and the option to be the same as in Figure 1. Figure 4 shows the wealth distribution when we have simulated 1000 realizations of the stock prices. The number of bins of the histograms is in all cases 25.

Panel (a) shows a histogram of the sample from the distribution of $e^{rT}H_0 - H_1$ and panel (b) shows a histogram of the sample from the distribution of ξX_1, where

$$X_1 = S_0 \left(\frac{S_1}{S_0} - e^{r\Delta t} \right). \tag{14}$$

and ξ is estimated using

$$\xi^* = \frac{\mathrm{Cov}(X_1, H_1)}{\mathrm{Var}(X_1)} = \frac{\mathrm{Cov}(S_1, H_1)}{\mathrm{Var}(S_1)}. \tag{15}$$

which gives $\xi = 0.38$. Panel (c) shows a histogram of the realizations of $e^{rT}H_0 + \xi X_1 - H_1$.

The mean over the 1000 realizations of the payout of the call is $EH_1 = E(S_T - K)_+ = 1.84$, the mean of the final wealth of delta hedging strategy is $EW_T = 0.84$. and the difference of these numbers is 0.99 (estimated price), which is close to the Black-Scholes price 1.193.

5 Examination

Possible questions in the examination:

13) Prove that the wealth process with self financing trading has the expression

$$W_t = W_0(1+r)^t + \sum_{k=0}^{t-1} \xi_{k+1}^T (1+r)^{t-k-1}[S_{k+1} - (1+r)S_k],$$

where W_0 is the initial wealth, $S_k = (S_k^1, \ldots, S_k^d)$ is the vector of stock prices, ξ_{k+1} is the vector of numbers of stock bought at time k, and we have available for trading the risk free bond which is increasing as

$$B_{k+1} = (1+r)B_k,$$

where $r > 0$ is the interest rate.

References

Bouchaud, J.-P. and Potters, M. (2003), *Theory of Financial Risks*, Cambridge University Press, Cambridge.