Lecture 10 and 11

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1 Derivative Pricing in a Binary Model

1.1 Binary Model

Let the market consist of stock S_t, bond B_t, and derivative H_t. We consider the single period model where $t = 0$ or $t = 1$. At time $t = 0$, the value of the stock is $S_0 = s_0$ and the value S_1 is a random variable. The stock may take only 2 values at time 1. At $t = 1$, the value is $s_{1,0}$ with probability $1 - p$ and the value is $s_{1,1}$ with probability p:

$$S_0 = s_0, \qquad P(S_1 = s_{1,0}) = 1 - p, \qquad P(S_1 = s_{1,1}) = p.$$

The bond has value b_0 at $t = 0$ and value $e^{r\Delta t} b_0$ at $t = 1$:

$$B_0 = b_0, \qquad B_1 = e^{r\Delta t} b_0.$$

Stock S_t is the underlying of derivative H_t. The derivative takes two possible values $H_1(s_{1,0})$ and $H_1(s_{1,1})$ and we have

$$P(H_1(S_1) = H_1(s_{1,0})) = 1 - p, \qquad P(H_1(S_1) = H_1(s_{1,1})) = p.$$

We want to find a fair value $H_0 \in \mathbf{R}$ for the the derivative at $t = 0$.

1.2 Solution

Let us consider the portfolio that has initial value H_0. At time 0 we buy bonds with the amount H_0, use the amount ξs_0 to buy the stock and sell short bonds of value ξs_0. At time 1 the value of the portfolio is

$$V_1 = e^{r\Delta t} H_0 + \xi(S_1 - e^{r\Delta t} s_0).$$

We can choose H_0 and ξ so that $V_1 = H_1(S_1)$ with probability one. The initial value of the portfolio is $V_0 = H_0$. Thus we get the price H_0 by solving the equations

$$
\begin{aligned}
e^{r\Delta t}H_0 + \xi(s_{1,0} - e^{r\Delta t}s_0) &= H_1(s_{1,0}),\\
e^{r\Delta t}H_0 + \xi(s_{1,1} - e^{r\Delta t}s_0) &= H_1(s_{1,1}).
\end{aligned}
$$

We get from the first equation

$$
H_0 = e^{-r\Delta t}H_1(s_{1,0}) - \xi(e^{-r\Delta t}s_{1,0} - s_0). \tag{1}
$$

Insertig this value of H_0 to the second equation gives

$$
\xi = \frac{H_1(s_{1,1}) - H_1(s_{1,0})}{s_{1,1} - s_{1,0}}\,.
$$

When we multiply the first equation with $1-p$, the second equation with p, and take the sum of the results, we get

$$
\begin{aligned}
H_0 &= e^{-r\Delta t}\left[(1-p)H_1(s_{1,0}) + pH_1(s_{1,1})\right]\\
&\quad -\xi\left[(1-p)(s_{1,0} - e^{r\Delta t}s_0) + p(s_{1,1} - e^{r\Delta t}s_0)\right]\\
&= e^{-r\Delta t}E_pH_1(S_1) - \xi E_p(S_1 - e^{r\Delta t}s_0).
\end{aligned}
$$

We can write also, using (1),

$$
\begin{aligned}
H_0 &= e^{-r\Delta t}\left[H_1(s_{1,0}) + \frac{H_1(s_{1,1}) - H_1(s_{1,0})}{s_{1,1} - s_{1,0}}\,(e^{r\Delta t}s_0 - s_{1,0})\right]\\
&= e^{-r\Delta t}\left[\frac{s_{1,1} - s_0e^{r\Delta t}}{s_{1,1} - s_{1,0}}\,H_1(s_{1,0}) + \frac{s_0e^{r\Delta t} - s_{1,0}}{s_{1,1} - s_{1,0}}\,H_1(s_{1,1})\right]\\
&= e^{-r\Delta t}\left[(1-q)H_1(s_{1,0}) + qH_1(s_{1,1})\right] &(2)\\
&= e^{-r\Delta t}E_qH_1(S_1), &(3)
\end{aligned}
$$

where

$$
q = \frac{s_0e^{r\Delta t} - s_{1,0}}{s_{1,1} - s_{1,0}}\,.
$$

2 Black-Scholes Price and Hedging

2.1 Approximation of Geometric Brownian Motion

In the 2nd lecture we have made an assumption that stock price S_T has a log-normal distribution. We extend this assumption to an assumption where S_t follows a geometric Brownian motion. We assume that

$$
S_t \sim S_0\exp\{\mu t + \sigma B_t\}, \qquad 0 \le t \le T,
$$

where B_t is the Brownian motion. The process B_t, $0 \leq t \leq T$, has the following properties:

1. $B_0 = 0$ with probability one,
2. $B_t \sim N(0,t)$,
3. $B_t - B_s$ is independent of B_s for $0 \leq s < t \leq T$.

We construct a discrete time process $S_{t_k,n}$, $k = 0, \ldots, n$, $t_k = kT/n$, which approximates the process S_t as $n \to \infty$. At start $S_{0,n} = S_0$ with probability one and conditionally on $S_{t-1} = s$ the process $S_{t,n}$ can go only up or down:

$$P\left(S_{t,n} = s(1+\sigma\sqrt{T/n}) \,|\, S_{t-1} = s\right) = \frac{1}{2} + \frac{\mu_0\sqrt{T/n}}{2\sigma} \tag{4}$$

and

$$P\left(S_{t,n} = s(1-\sigma\sqrt{T/n}) \,|\, S_{t-1} = s\right) = \frac{1}{2} - \frac{\mu_0\sqrt{T/n}}{2\sigma}\,, \tag{5}$$

where

$$\mu_0 = \mu + \frac{1}{2}\sigma^2.$$

Thus we can write $S_{t_k,n}$ as

$$S_{t_k,n} = S_0 \prod_{i=1}^{k} \left(1 + w_i\sigma\sqrt{T/n}\right),$$

where w_i are i.i.d. random variables with

$$P(w_i = 1) = \frac{1}{2} + \frac{\mu_0\sqrt{T/n}}{2\sigma}, \qquad P(w_i = -1) = \frac{1}{2} - \frac{\mu_0\sqrt{T/n}}{2\sigma}.$$

Now

$$Ew_i = \frac{\mu_0\sqrt{T/n}}{\sigma}, \qquad Ew_i^2 = 1, \qquad \mathrm{Var}(w_i) = 1 - \left(\frac{\mu_0\sqrt{T/n}}{\sigma}\right)^2.$$

We can write

$$\begin{aligned} \log_e\left(\frac{S_{t_k,n}}{S_0}\right) &= \sum_{i=1}^{k} \log_e\left(1 + w_i\sigma\sqrt{T/n}\right) \\ &\approx \sigma\sqrt{T/n}\sum_{i=1}^{k} w_i - \frac{1}{2}\frac{\sigma^2 T}{n}\sum_{i=1}^{k} w_i^2, \end{aligned}$$

because $\log(1+x) \approx x - \frac{1}{2}x^2$. Thus

$$E\left(\log_e\left(\frac{S_{t_k,n}}{S_0}\right)\right) \approx \left(\mu_0 - \frac{1}{2}\sigma^2\right) t_k = \mu t_k$$

and

$$\mathrm{Var}\left(\log_e\left(\frac{S_{t_k,n}}{S_0}\right)\right) \approx \sigma^2 t_k,$$

where we used the approximation $\mathrm{Var}(w_i) \approx 1$ for large n. Since w_i are independent, we get by the central limit theorem for large n the approximation

$$\log_e\left(\frac{S_{t_k,n}}{S_0}\right) \sim \mu t_k + \sigma B_{t_k}, \tag{6}$$

where B_t is the Brownian motion.

2.2 Backward Induction

We can use the single period model to find the price of an option at time 0. We know the price at time n. We can use the single period model to calculate the price at time $n-1$, and go backwards step by step to obtain the price at time 0. The evolution of the price has been described with a recombining binary tree. At time k, $0 \le k \le n$, there are $k+1$ possible prices. Let us denote those prices as $s_{k,j}$, $k = 0,\ldots,n$, $j = 0,\ldots,k$. When we are at price $s_{k-1,j}$, there are two possible prices $s_{k,j}$ and $s_{k,j+1}$ for the stock. These prices are obtained using (4) and (5) as

$$s_{k,j+1} = s_{k-1,j}(1 + \sigma\sqrt{T/n}), \qquad s_{k,j} = s_{k-1,j}(1 - \sigma\sqrt{T/n}).$$

1. When we are at time $n-1$, and the stock price is $s_{n-1,j}$, then we know the two possible prices for the derivative: $H_n(s_{n,j})$ and $H_n(s_{n,j+1})$. We can use the single period model to calculate the price at time $n-1$. We get the price from (3) as

   $$H_{n-1}(s_{n-1,j}) = (1 + r\Delta t)\left[(1-q)H_n(s_{n,j}) + qH_n(s_{n,j+1})\right],$$

   where

   $$q = \frac{s_{n-1,j}(1 + r\Delta t) - s_{n,j}}{s_{n,j+1} - s_{n,j}} = \frac{1}{2} + \frac{r\sqrt{T/n}}{2\sigma},$$

   because $\Delta t = T/n$.

2. For $k = 1, \dots, n$, when we are at time $k-1$, at price $s_{k-1,j}$, then we know from the previous calculation the two possible prices for the derivative: $H_k(s_{k,j})$ and $H(s_{k,j+1})$. We can use the single period model to calculate the price at time $k-1$. We get the price from (3) as

$$H_{k-1}(s_{k-1,j}) = (1 + r\Delta t)\left[(1-q)H_k(s_{k,j}) + qH_k(s_{k,j+1})\right]. \qquad (7)$$

Inductively we get the price at 0 as

$$H_0(s_0) = (1 + r\Delta t)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} H_n(s_{n,j}), \qquad (8)$$

where the possible prices of the stock at the expiration are

$$s_{n,j} = a^j (2-a)^{n-j} s_0, \qquad j = 0, \dots, n.$$

with

$$a = \left(1 + \sigma\sqrt{\frac{T}{n}}\right).$$

The price in (8) is the expectation with respect to a binomial distribution. The recombining binary tree is the same as previously but the probabilities in (4) and (5) are now replaced by

$$P\left(S_{t,n} = sa \mid S_{t-1} = s\right) = q, \qquad P\left(S_{t,n} = s(2-a) \mid S_{t-1} = s\right) = 1 - q$$

We can use the same argument as was used to derive (6) to conclude that the price in (8) can be approximated by an expectation with respect to the log-normal distribution: we have

$$H_0(s_0) = e^{-rT} E H_T(X) = e^{-rT} \int_0^\infty H_T(x) f_X(x)\, dx,$$

where $f_X : (0,\infty) \to \mathbf{R}$ is the density of random log-normal variable X, that has the distribution

$$\log_e X - \log_e s_0 \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right).$$

2.3 Black Scholes Hedging

The Black Scholes price of the call option with strike price K, when the stock price is S_t at t, and the maturity is T, is equal to

$$C_t(S_t, K, T) = S_t \Phi(z_+) - K e^{-r(T-t)} \Phi(z_-),$$

where

$$z_{\pm} = \frac{\log_e(S_t/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

where Φ is the distribution function of the standard Gaussian distribution and σ is the volatility parameter. We have

$$\frac{\partial C_t(S,K,T)}{\partial S} = \Phi(z_+),$$

since

$$\frac{\partial C_t(S,K,T)}{\partial S} = \Phi(z_+) + e^{-r(T-t)}\left[Se^{r(T-t)}\,\frac{\partial \Phi(z_+)}{\partial S} - K\,\frac{\partial \Phi(z_-)}{\partial S}\right],$$

$$\frac{\partial \Phi(z_+)}{\partial S} = \phi(z_+)\cdot\frac{\partial z_+}{\partial S} = \phi(z_+)\cdot\frac{1}{\sigma\sqrt{T-t}}\cdot\frac{K}{S},$$

$$\frac{\partial \Phi(z_-)}{\partial S} = \phi(z_-)\cdot\frac{1}{\sigma\sqrt{T-t}}\cdot\frac{K}{S},$$

and finally

$$e^{r(T-t)}\phi(z_+) = \frac{K}{S}\,\phi(z_-).$$

3 Examination

Possible questions in the examination:

14) (a) Define the single step binary model for stock evolution.

(b) Find the price of an European option in this model.