Lecture 6

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1 Risk measures

1.1 Conditional variance estimation

The variance of Y can be written as

$$\mathrm{Var}(Y) = EY^2 - (EY)^2.$$

Thus the conditional variance can be written as

$$\mathrm{Var}(Y \mid X = x) = E(Y^2 \mid X = x) - [E(Y \mid X = X)]^2$$

and thus the kernel estimator of the conditional variance can be defined as

$$\hat{f}(x) = \sum_{i=1}^{n} p_i(x)\, Y_i^2 - \left(\sum_{i=1}^{n} p_i(x)\, Y_i \right)^2 ,$$

where

$$p_i(x) = \frac{K_h(x - X_i)}{\sum_{i=1}^{n} K_h(x - X_i)}, \qquad i = 1, \ldots, n, \tag{1}$$

$K : \mathbf{R}^d \to \mathbf{R}$ is the kernel function, $K_h(x) = K(x/h)/h^d$, and $h > 0$ is the smoothing parameter.

1.2 Conditional density estimation

Kernel estimator A kernel estimator of the density of Y, based on data $Y_1, \ldots, Y_n$, is defined as

$$\tilde{f}_Y(y) = \frac{1}{n} \sum_{i=1}^{n} L_g(y - Y_i), \qquad y \in \mathbf{R},$$

where $L : \mathbf{R} \to \mathbf{R}$ is the kernel function, $L_g(y) = L(y/g)/g$, and $g > 0$ is the smoothing parameter. A kernel estimator of the conditional density of Y given X, associated with the kernel regression function estimator, is defined as

$$\tilde{f}_{Y|X}(y \,|\, X = x) = \sum_{i=1}^{n} p_i(x) L_g(y - Y_i), \qquad y \in \mathbf{R}, \tag{2}$$

where the weights $p_i(x)$ are defined in (1).

Histogram estimator A histogram estimator of the density of Y, based on data $Y_1, \ldots, Y_n$, is defined as

$$\tilde{f}_Y(y) = \sum_{R \in \mathcal{P}} \frac{n_R/n}{\mathrm{volume}(R)} I_R(y) \qquad y \in \mathbf{R},$$

where $\mathcal{P}$ is a partition of $\mathbf{R}$ and

$$n_R = \#\{i : Y_i \in R,\ i = 1, \ldots, n\}$$

is the number of observations in R. A histogram estimator of the conditional density of Y given X, associated with a kernel regression function estimator, is defined as

$$\tilde{f}_{Y|X}(y \,|\, X = x) = \sum_{R \in \mathcal{P}} \frac{n_R(x)/n}{\mathrm{volume}(R)} I_R(y) \qquad y \in \mathbf{R},$$

where

$$n_R(x) = n \cdot \sum_{i: Y_i \in R} p_i(x),$$

and $p_i(x)$ are defined in (1).

1.3 Conditional distribution function estimation

The distribution function of $Y \in \mathbf{R}$ is defined as

$$F_Y(y) = P(Y \leq y), \qquad y \in \mathbf{R}.$$

The distribution function can be estimated with the empirical distribution function, which is defined as

$$\hat{F}_Y(y) = \frac{1}{n} \sum_{i=1}^{n} I_{(-\infty, y]}(Y_i) = n^{-1} \#\{i : i = 1, \ldots, n, Y_i \leq y\}. \tag{3}$$

The conditional distribution function of Y given X is defined as

$$F_{Y|X}(y \,|\, X = x) = P(Y \le y \,|\, X = x).$$

We can define the kernel estimator of the conditional distribution function as

$$\hat{F}_{Y|X}(y \,|\, X = x) = \sum_{i=1}^{n} p_i(x) I_{(-\infty,y]}(Y_i), \qquad (4)$$

where $p_i(x)$ are the kernel weights defined in (1).

1.4 Conditional quantile estimation

An estimator of a quantile of Y can be defined with the help of the empirical distribution function $\hat{F}_Y(y)$, defined in (3), as

$$\begin{aligned}\hat{Q}_p(Y) &= \inf\{y : \hat{F}_Y(y) \ge p\}, \\ &= \begin{cases} Y_{min}, & 0 < p \le 1/n \\ Y_{min+1}, & 1/n < p \le 2/n \\ \cdots \end{cases}\end{aligned}$$

where $0 < p < 1$ and

$$\begin{aligned} Y_{min} &= \min\{Y_1, \ldots, Y_n\} \\ Y_{min+1} &= \min(\{Y_1, \ldots, Y_n\} \setminus \{Y_{min}\}) \\ &\vdots \end{aligned}$$

The kernel estimator of the conditional quantile is

$$\begin{aligned}\hat{Q}_p(Y \,|\, X = x) &= \inf\{y : \hat{F}_{Y|X}(y \,|\, X = x) \ge p\} \\ &= \begin{cases} Y_{min}, & 0 < p \le p_1(x) \\ Y_{min+1}, & p_1(x) < p \le p_1(x) + p_2(x) \\ \cdots \end{cases}\end{aligned}$$

where $\hat{F}_{Y|X}(y \,|\, X = x)$ is the estimator of the conditional distribution function defined in (4).

2 Illustrations

We look at the following code in

http://cc.oulu.fi/~jklemela/finatool/

```
# we obtain returns of the DAX stock index

ticker<-c("^GDAXI")
destfile<-"~/pois"
ry<-read.yahoo(ticker, source="web", destfile=destfile)
#save(file="/home/jsk/Arti/statfina/Dax.var",list=c("ry"))
#load(file="/Users/jsk/Karhu/Arti/statfina/DaxMdax.var")
dm<-data.manip(ry,ticker)
method<-"return"
S<-returns(dm$data,method=method)
n<-length(S)
plot(S,type="l")

# we calculate volatilities for the 5 day periods

perlen<-5
pernum<-floor(n/perlen)
volas<-matrix(0,pernum,1)
for (i in 1:pernum){
    beg<-(i-1)*perlen+1
    end<-(i-1)*perlen+perlen
    period<-S[beg:end]
    volas[i]<-sqrt(sum(period^2)/perlen)*sqrt(252)
}
plot(volas,type="l")

# we make a logarithmic transform for the x-variable

x<-log(dendat[,1])
y<-dendat[,2]
plot(x,y)

h<-0.2
N<-80
pcf<-pcf.kernesti(x,y,h,N)
dp<-draw.pcf(pcf)
matplot(dp$x,dp$y,type="l",xlim=c(-3.5,0.2),ylim=c(0,1.2))
matplot(x,y,add=TRUE)

# we look at the conditional densities using kernel estimates
```

```
hd<-0.04
N<-100
pcf<-pcf.kerndens(matrix(y,length(y),1),hd,N,kernel="gauss")
dp<-draw.pcf(pcf)
plot(dp$x,dp$y,type="l",xlab="y",ylab="")

arg<-1
kw<-kernesti.weights(arg,x,h)

pcf2<-pcf.kerndens(matrix(y,length(y),1),hd,N,kernel="gauss",weights=kw)
dp2<-draw.pcf(pcf2)
plot(dp2$x,dp2$y,type="l")

minx<-0
maxx<-1.2
miny<-0
maxy<-10
plot(x="",y="",xlim=c(minx,maxx),ylim=c(miny,maxy),xlab="",ylab="")
matplot(dp$x,dp$y,type="l",add=TRUE)
matplot(dp2$x,dp2$y,type="l",add=TRUE,col="blue")



# conditional densities in portfolio selection

ticker<-c("^GDAXI","^MDAXI")
destfile<-"~/pois"
ry<-read.yahoo(ticker, source="web", destfile=destfile)
dm<-data.manip(ry,ticker)

method<-rep("relative",length(ticker))
U<-returns(dm$data,method=method)

marginal<-"gauss"
k<-4
mp<-make.portdat(U,k,marginal=marginal) #,rate=0.02/360)

h<-1
binlkm<-5
plot.condi(Z=mp$Z,arg=mp$arg,X=mp$X,h=h,binlkm=binlkm)
```

```
plot.condi.seq(U,k=k,h=h)
```

3 Examination

A possible questions in the examination:

5) Explain how conditional variance and conditional quantiles can be estimated with a kernel method.