Non-linear partial differential equations

Project description

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1. BACKGROUND

The $p$-Laplace operator, perhaps the simplest non-linear generalization of the Laplace operator, has been thoroughly studied during the last decades. There is now a very satisfactory theory for this operator, however, with the drawback that it is very dependent on the highly special form of the operator, and does not readily generalize to new settings. In our research we will consider more general differential operators built on the $p$-Laplace operator – we aim at developing a theory that will eventually shed more light also in the basic non-linear case. Applications in several branches of modern analysis, like calculus of variations, non-linear potential theory and the theory of quasiconformal mappings, attest to the centrality of these kind of operators.

By way of motivation we remind the reader that the $p$-Laplace operator occurs for instance in the variant of Navier–Stokes that describes the motion of non-Newtonian fluids. These fluids, then, are such that the velocity gradient depends in a non-linear way on the stress tensor; they occur for instance in glaceology, rheology, non-linear elasticity and flow through a porous medium. Other applications include image processing and mean curvature flow.

The following are the three main aspects of previous studies that we want to reconsider.

1. Classical studies have focused on stationary equations, i.e. time has been frozen to a fixed moment. These equations are elliptic in nature. In order to understand the dynamics of the equations (and fluids) we need to consider also the time evolution, which leads to parabolic equations. Very little is known about the behavior of these equations.

2. Certain non-Newtonian fluids, e.g. electrorheological fluids, feature a more complicated, so-called non-standard growth in the highest order differential term. In order to understand the behavior of these kind of (models of) fluids, it is necessary to understand the effect of the exponent $p$ in a local sense.

3. The $p$-Laplace equation has been studied at length in Euclidean spaces, sometimes with weights. In order to unify the treatment and prove results that are valid in a greater range of situations (e.g. manifolds, graphs) it is natural to conduct the research in the framework of metric measure spaces.

In the next three sections we describe in more detail our research which focuses on each of these three areas. Although it is necessary to make such a division for clarity of presentation, it does not do justice to the fact that all of these lines of investigation are profoundly connected through the fact that they all strive to understand stability of (results about) the $p$-Laplacian under “perturbations”. These connections are clearly displayed in a few sections, such as the sections “Doubling and Poincaré” (which features a somewhat unexpected relation between analysis on metric spaces and parabolic partial differential equations) and the section “Equations with non-standard growth”.

2. DEGENERATE PARABOLIC EQUATIONS

The principal prototype of the equation that we have in mind is the $p$-parabolic equation

\[
\frac{\partial u}{\partial t} = \text{div}(|Du|^{p-2}Du), \quad 1 < p < \infty,
\]
which is a time dependent counterpart of the \( p \)-Laplace equation. In contrast to the elliptic case, the theory for the parabolic equation is to a large extent open. Indeed, there are several phenomena and difficulties which are not present in the elliptic case. For example, disturbances have finite speed of propagation if \( p > 2 \) and when \( 1 < p < 2 \) there is an extinction phenomenon in finite time. These are often natural properties in applications.

Another prototype is the doubly non-linear equation

\[
\frac{\partial}{\partial t} (|u|^{p-2} u) = \text{div}(|Du|^{p-2} Du), \quad 1 < p < \infty,
\]

and its variants.

We also study more general parabolic equations of the type

\[
\frac{\partial u}{\partial t} = \text{div}(|Du|^{p(x,t)-2} Du).
\]

We hope that it will be possible to develop methods for the problems described in the previous section which will allow us to cover also the variable exponent case.

**Harnack estimates.** It is known that a scale invariant parabolic Harnack inequality holds for the doubly non-linear equation. This is not the case for the \( p \)-parabolic equation, since examples show that Harnack’s inequality must have an intrinsic formulation which depends on the solution itself. To describe this fascinating phenomenon, we mention that the fundamental solution to (1), first constructed by Barenblatt, is non-negative and compactly supported in the space variable at any given moment in time.

For bounded coefficients, it is known that weak solutions of the equation

\[
\frac{\partial u}{\partial t} = \text{div} A(x, t, Du), \quad 1 < p < \infty,
\]

are locally Hölder continuous. Very recently DiBenedetto, Gianazza and Vespri made a breakthrough by proving Harnack type estimates for the bounded coefficient case. The purpose of this research is to study whether this method can be combined with our own ideas to prove Harnack type estimates for super- and subsolutions as well. These estimates should be useful in proving boundary continuity.

**Doubling and Poincaré.** Another question on Harnack type estimates is related to analysis on metric spaces (which will be described in more detail below). The doubling condition and the Poincaré inequality are rather standard assumptions in analysis on metric spaces. Grigor’yan and Saloff-Coste observed that doubling condition and the Poincaré inequality are not only sufficient but also necessary conditions for a scale invariant parabolic Harnack principle for the heat equation on Riemannian manifolds. The sufficiency has been shown by Kinnunen and Kuusi (2007) for the doubly non-linear equation (2) in a Euclidean space. We aim at proving necessity also in this more general setting. This seems to require estimates for a caloric potential which is a parabolic counterpart of the Wolff potential in the elliptic case.

**Higher integrability results.** In our previous studies we were able to establish local higher integrability properties for the gradients of solutions of (3). Our objective is to apply these results in global higher integrability questions in nonsmooth domains and other regularity problems for systems of \( p \)-parabolic type. In elliptic theory, higher
integrability is an essential ingredient in proving regularity results for systems and our aim is to create a similar theory in the parabolic case.

**Boundary continuity.** There are also many other problems related to regularity theory in the parabolic case. For example the continuity of a solution up to the boundary seems to be completely open for \( p \neq 2 \). In the quadratic case \( p = 2 \) for the heat equation this has been studied by Evans and Gariepy. They show that a necessary and sufficient condition is given by so-called Wiener criterion.

3. **Equations with non-standard growth conditions**

In an effort to model various non-homogeneous materials, especially electrorheological fluids, many investigators have recently been interested in differential equations with non-standard growth conditions. This leads naturally to a generalization of Lebesgue and Sobolev spaces where the exponent is allowed to vary. From a mathematical point-of-view, these spaces are challenging because many techniques, notably convolution, rearrangement and the co-area formula, are not available in variable exponent setting.

**Non-linear potential theory.** In some regards the variable exponent theory is lagging behind the fixed exponent case. An example is elliptic non-linear potential theory. For instance, there are some problems with the definition of harmonic and superharmonic functions. It was found by Harjulehto, Kinnunen and Lukkari (2007) that the variable exponent Harnack inequality shows several interesting new facets: the inequality is of the form

\[
\sup_{x \in B} u(x) \leq c \inf_{x \in B} u(x) + \text{diam} B,
\]

where, furthermore, \( c \) depends on the norm of \( u \)! Moreover, these deviations from the classical form are to some extent necessary, as shown by examples. In some sense we see that we again run into some kind of intrinsic Harnack inequality. We have also investigated foundational properties of harmonic and superharmonic functions and continue to pursue this direction further.

**Scales of function spaces.** Another area where there is much catching up to do has to do with other function spaces (e.g. Besov or Triebel–Lizorkin spaces) with variable exponent. In this regard we also hope to incorporate spaces of variable smoothness studied by Besov. This work is based on an paper by Diening and Hästö (2005) on variable exponent trace spaces. These spaces are interesting for a variety of reasons. First, trace spaces play a role in the study of admissible boundary values related to a differential equations. Besov spaces provide the natural scale for embeddings between these spaces. Second, Besov spaces provide the natural scale for embeddings between these spaces. Third, these scales might clarify the roles of the extremes of the scale of Lebesgue and Sobolev spaces. For instance we know that it is sometimes natural to work with the Hardy space \( H^1 \) instead of the Lebesgue space \( L^1 \) at the lower end of the Lebesgue spectrum.

One very interesting question related to the low end of the spectrum of Sobolev spaces (Harjulehto, Hästö and Latvala, 2006): Suppose that \( p \to 1 \) in some parts of the domain. What happens to fundamental properties, like existance of a Dirichlet energy integral minimizer, in this case? This question is also central to applications...
in image processing (as described by Levine, Chen and Rao, 2006), since \( p = 1 \) corresponds to total variation smoothing, which must definitely be handled by the model.

4. **Partial differential equations on metric measure spaces**

In the Euclidean case solutions of the \( p \)-Laplace equation can be defined as minimizers of the variational integral

\[
\int |Du|^p \, dx.
\]

The minimizers are solutions to the corresponding Euler-Lagrange equation, which in this case is the \( p \)-Laplace equation. It is not clear what the counterpart for the \( p \)-Laplace equation is in a general metric measure space, but the variational approach is available. Direct methods in the calculus of variations can be applied to prove the existence of the Dirichlet problem. Kinnunen and Shanmugalingam (2001) shown that minimizers satisfy Harnack’s inequality, the strong maximum principle, and are locally Hölder continuous, under standard assumptions on the space.

**Regularity of quasiminimizers.** Our research group focuses on the following aspects of the regularity theory for minimizers (or quasiminimizers) of (4) on a metric measure space. The first aim is to extend the local and global higher integrability results to the metric setting. As an application we study stability with respect to the exponent. This is closely related to the variable exponent case which was described above.

The second aspect is to study the Moser iteration scheme for quasiminimizers of (4). It has already been shown that Moser’s method applies to minimizers and to a large extent to quasiminimizers, but there is still a question about a logarithmic estimate for quasiminimizers which should be solved in order to be able to obtain the full result. In addition, parabolic quasiminimizers seem to open a possible route to study solutions of (3) in a metric measure space.

**Doubling and Poincaré.** One of the main goals of our research is to better understand these assumptions. In addition to the connection with the parabolic Harnack inequality, we shall focus on the following question. It is known that if the space in complete, then these assumptions imply that the space has to be quasiconvex. This means that every pair of points can be joined by a curve whose length is bounded by the distance between the points. Our purpose is to obtain quantitative versions of this result, which give some kind of estimates on the number of the curves. Moreover, we study characterizations of Sobolev-Poincaré inequalities in terms of capacities. Particular attention will be paid the case when the gradient is integrable to the power one. Very little is known about this case so far.

5. **Research team**

Our group consists of two senior researchers and 10 graduate students.

We are currently running two seminars, one at the University of Oulu and one at Helsinki University of Technology. There is also extensive interaction between the seminars, with students traveling between the seminar to present their work. We will
continue to nurture these and other national co-operation, also in connection with the national Graduate School of Analysis.

We rely heavily on group work to carry out our extensive research program. Although much collaboration will take place within the group, we also work at length with other colleagues in Finland and abroad.

Our main contacts in Finland are at
- Helsinki University of Technology
- University of Helsinki
- University of Joensuu
- University of Jyväskylä

We also have extensive contacts with researchers abroad:
- Czech Republic: Prague
- Germany: Freiburg
- India: Chennai
- Italy: Firenze, Napoli, Parma, Pavia
- Norway: Bergen, Trondheim
- Poland: Warsaw
- Portugal: Algarve, Aveiro, Coimbra
- Russia: St. Petersburg
- Spain: Barcelona, Madrid, Sevilla
- Sweden: Linköping, Umeå
- USA: Ann Arbor, Ames, Cincinnati, Columbia University, Kentucky

We will invite several collaborators to visit Finland each year. It is also important that the graduate students have the opportunity for both conference participation and extended visits at our foreign collaborators. Three of the graduate students have already spent extended periods in Cincinnati, Linköping and New York, and both senior researchers have spent several years working abroad.