Nordic Mathematical Team Contest 2011

Competition time February 3, 12.00 CET – February 7, 18.00 CET.

Solutions Solutions must be written in English. The preferred format is a `pdf` document compiled from a LaTeX source. Scanned solutions written by hand are acceptable, but the jury is at a liberty to deduct points for illegibility.

Submission Solutions should be submitted to the chairman of the jury at:

`qimh@math.su.se`

Questions Questions regarding the formulation of the problems may be directed towards the chairman of the jury. *Answers will be posted on the official website.*

Score Each problem is worth 6 points.

1. Find all smooth real functions y such that, for any x, the numbers $y(x)$, $y'(x)$, $y''(x)$, ... are in

   (a) arithmetic progression. (3 points)

   (b) geometric progression. (3 points)

2. Let p be a real polynomial of degree $n \geq 2$. Denote by $a_1 < a_2 < \ldots$ the real zeroes of p, and by $b_1 < b_2 < \ldots$ the zeroes of p'. For what n does the inequality

   $$b_1 - a_1 < a_2 - b_1$$

   always hold?

3. There are $n \geq 2$ glasses of volume 1. One glass contains juice concentrate, and the others water. You may pour any quantity of liquid from one glass to another. The liquids mix completely. After a finite number of operations, each glass must be either empty or have the same concentration as the other non-empty ones.

   What is the maximum volume of juice (water plus concentrate) you can get at the end?

4. Given a sequence z_n of complex numbers converging to 0, is it always possible to choose signs $a_n \in \{-1, +1\}$ in such a way that

$$\sum_{n=0}^{\infty} a_n z_n$$

converges?

5. Prove that the $n \times n$ determinant

$$\begin{vmatrix} a & b & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ c & a & b & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & c & a & b & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & c & a & b \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & c & a \end{vmatrix} = \prod_{k=1}^{n} \left(a - \sqrt{bc} \cos \frac{k\pi}{n+1} \right).$$

6. Six distinct lines $L_1, L_2, L_3, M_1, M_2, M_3$ in space intersect in a point. The lines L_i are pairwise orthogonal, as are the lines M_j. Prove that the six lines lie on a (quadratic) cone.

7. Let a_n, for $n \in \mathbf{N}$, denote the unique solution in the interval $(n, n+1)$ of the equation

$$\tan \pi x = \frac{1}{x}.$$

Does the series

$$\sum_{n=0}^{\infty} (a_n - n)$$

converge?

8. Let

$$w(e^{it}) = \sum_{k=-n}^{n} c_j e^{kit}$$

be a trigonometric polynomial whose values are real and non-negative for all real t. Prove that there is a complex polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ such that

$$w(e^{it}) = |p(e^{it})|^2.$$

9. (a) Show that for each $k \geq 1$ there exists a positive constant C such that

$$\int_1^n \frac{(t-1)^{n-1}}{t^{n+k}} \, dt \leq \frac{C}{n^k}, \quad \text{for any } n \in \mathbf{N}.$$

(2 points)

(b) Let $f : [1, +\infty) \to (0, 1]$ be a decreasing continuous function that satisfies:

$$f(1) = 1, \quad \lim_{t\to\infty} f(t) = 0, \quad f(2t) \geq a f(t),$$

for some $a > 0$. Prove that there exists a positive constant C such that

$$\int_1^n \frac{(t-1)^{n-1} f(t)}{t^{n+1}}\, dt \leq \frac{C f(n)}{n}, \quad \text{for any } n \in \mathbf{N}.$$

(4 points)

10. Evaluate the sum

$$\sum_{k=1}^{2010} \mathrm{GCD}(k, 2010) \cos \frac{2\pi k}{2010}.$$

11. Consider the following system: Points A and E are fixed. Points B, C, and D are allowed to move in the plane in such a way that

$$|AB| = |BC| = |CD| = |DE| = 1.$$

Describe (the topology of) the configuration space

$$\{(x_B, y_B, x_C, y_C, x_D, y_D) \in \mathbf{R}^6\}$$

when

(a) $A = (0, 0)$ and $E = (3.9, 0)$. (2 points)

(b) $A = (0, 0)$ and $E = (1, 0)$. (4 points)