ON BAKER TYPE LOWER BOUNDS FOR LINEAR FORMS

TAPANI MATALA-AHO

ABSTRACT. A criterion is given for studying (explicit) Baker type lower bounds of linear forms in numbers $1, \Theta_1, \ldots, \Theta_m \in \mathbb{C}^*$ over the ring $\mathbb{Z}_I$ of an imaginary quadratic field $I$. This work deals with the simultaneous auxiliary functions case.

1. Introduction

We give a criterion for studying (explicit) Baker type lower bounds of linear forms in given numbers $\Theta_0, \ldots, \Theta_m \in \mathbb{C}^*$. Throughout this work, let $I$ denote an imaginary quadratic field with $\mathbb{Z}_I$ it’s ring of integers. By an explicit Baker type lower bound we mean any positive lower bound

\begin{equation}
|\beta_0 \Theta_0 + \ldots + \beta_m \Theta_m| > F(H_0, \ldots, H_m, m)
\end{equation}

valid for all $\mathbf{\beta} = (\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}_I^{m+1} \setminus \{\mathbf{0}\}$ with $\prod_{j=0}^{m} H_j \geq H \geq 1$, $H_j \geq h_j = \max\{1, |\beta_j|\}$, where the dependence on each individual term $H_0, \ldots, H_m$, $m$ and numbers $\Theta_0, \ldots, \Theta_m$ is explicitly given in the functional dependence $F(H_0, \ldots, H_m, m)$ and the dependence on $\Theta_0, \ldots, \Theta_m, m$ is explicitly given in the constant $H = H(\Theta_0, \ldots, \Theta_m, m)$.

With the assumption that $\gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{Q}^*$ are distinct, Baker [1] proved that there exist positive constants $\delta_1, \delta_2$ and $\delta_3$ such that

\begin{equation}
|\beta_0 e^{\gamma_0} + \ldots + \beta_m e^{\gamma_m}| > \frac{\delta_1 M^{1-\delta(M)}}{\prod_{j=0}^{m} h_j},
\end{equation}

for all $\mathbf{\beta} = (\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$, $h_j = \max\{1, |\beta_j|\}$, with

\begin{equation}
\delta(M) \leq \frac{\delta_2}{\sqrt{\log \log M}}, \quad M = \max_{0 \leq j \leq m} \{|\beta_j|\} \geq \delta_3 > e.
\end{equation}

Here we note that the constants $\delta_1, \delta_2, \delta_3$ in Baker’s work [1] are not explicitly given. Mahler [9] made Baker’s result completely explicit.

There are many subsequent works, where the authors prove Baker type lower bounds for values of functions belonging usually to a class of Siegel’s $E$- or $G$-functions or $q$-hypergeometric functions evaluated at rational points, see e.g. [5], [6], [13] and [14]. For a more comprehensive list of references, see [6]. In the above mentioned works Siegel’s lemma is a standard tool for producing a first or second kind Padé-approximation construction of certain auxiliary functions. These constructions correspond to one linear form

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(one auxiliary function) or simultaneous linear forms (several auxiliary functions).

In this work we shall not do such constructions but we are interested in the next step. Namely, how to use appropriate linear forms to prove Baker type lower bounds? We shall answer the above question by giving a criterion in the simultaneous linear forms case.

Let us describe our criterion in a nutshell. Fix \( \Theta_1, \ldots, \Theta_m \in \mathbb{C}^* \) and put \( \pi = (n_1, \ldots, n_m)^T \), \( N = N(\pi) = n_1 + \ldots + n_m \). Assume that we have a sequence of simultaneous linear forms

\[
L_{k,j}(\pi) = A_{k,j}(\pi), \quad k = 0, 1, \ldots, m, \quad j = 1, \ldots, m, \quad \pi \in \mathbb{Z}^m_+,
\]

where \( A_{k,j} = A_{k,j}(\pi) \in \mathbb{Z}_l \) satisfy a certain determinant condition. Suppose also that

\[
|A_{k,0}(\pi)| \leq e^{(aN+b \log N) \Theta_1 + b_0N(\log N)^{1/2} + b_1N + b_2 \log N + b_3},
\]

\[
|L_{k,j}(\pi)| \leq e^{(dN-c \log N) \Theta_1 + c_0N(\log N)^{1/2} + c_1N + c_2 \log N + c_3},
\]

for \( k, j = 0, 1, \ldots, m \), where \( a, b, c, d, b_i, c_i \) are non-negative parameters satisfying \( a, c - dm > 0 \). Then, in the cases \( g(N) \in \{1, \log N, N\} \), we shall prove that there exist explicit positive constants \( F_l, G_l (l \in \{1, 2, 3\}) \), such that

\[
|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| > F_l \left( \prod_{j=1}^m (2mH_j) \right)^{-\frac{a}{dm} - \epsilon_i(H)}
\]

holds for all \( \beta = (\beta_0, \beta_1, \ldots, \beta_m)^T \in \mathbb{Z}_l^{m+1} \backslash \{0\} \) and \( H = \prod_{j=1}^m (2mH_j) \geq G_l \), \( H_j \geq h_j = \max\{1, |\beta_j|\} \) with an error term \( \epsilon_i(H) \rightarrow 0 \) as \( H \rightarrow \infty \). The constants \( F_l, G_l \) and the error term will be given explicitly in terms of the parameters \( a, b, c, d, b_i, c_i \) and, in particular of \( m \).

The underlying idea behind our treatment is well known already from Baker’s work [1]. Namely, the idea, see formula (22) in [1], is to fix the parameter \( n_j \) with the corresponding individual height \( H_j \) (in our notation). In our work we shall express this phenomenon first in a nutshell, see (4.10), and then in a refined form, see (4.14).

An advantage of our treatment compared with existing treatments is that one may easily see if the contribution to the lower bound is coming from the Diophantine method itself or from the auxiliary construction. For example, apart from the condition \( n_1 + \ldots + n_m = N \), we don’t need any extra condition between \( n_j \) and \( N \) in our treatment. Of course, some extra conditions may be needed for good auxiliary constructions. In particular, this is the case when Siegel’s lemma is involved. See, e.g. [13], formula (14), where the authors additionally assume that \( n_j > \delta N, j = 1, \ldots, m \), for some \( 0 < \delta < 1/m \). In [10] the corresponding condition reads \( n_j > 2N/(\log N) \), \( j = 1, \ldots, m \), formula (4) in Chapter III. In [4], however, you may find a slightly different approach.

Our Theorems 3.2, 3.4 and 3.6 are designed to be applied in the following manner. Let \( f(z) \) be a \( G^- \), \( E^- \) or \( q \)-hypergeometric function and denote \( \Theta_1 = \)}
$f(\alpha_1), ..., \Theta_m = f(\alpha_m)$, $\alpha_1, ..., \alpha_m \in \mathbb{I}^*$.

Suppose that one can construct simultaneous linear forms of the type (1.4) satisfying the estimates (1.5) and (1.6) with a certain determinant condition, then our Theorem 3.2, 3.4 or 3.6 will give a corresponding Baker type lower bound (1.7). So far our results (Theorems 3.4 and 3.6) have been applied in the works [4] and [8]. In [4], Ernvall-Hytonen, Leppälä and Matala-aho constructed simultaneous linear forms of the type (1.4) (satisfying conditions (1.5)-(1.6) with $g(N) = \log N$) for the exponential function values $e^{\alpha_0}, e^{\alpha_1}, ..., e^{\alpha_m}$, where $\alpha_0, ..., \alpha_m \in \mathbb{I}$. (Note that the exponential function belongs to the class of Siegel’s $E$-functions.) By applying Theorem 3.4 of the present paper the authors in [4] proved substantial improvements of the explicit versions, see Mahler [9] and Sankilampi [10], of Baker’s work [1] about exponential values at rational points. In particular, the dependence on $m$ is improved. As an example from the work [4] we mention a new explicit Baker type lower bound

$$|\beta_0 + \beta_1 \Theta + \beta_1 e^2 + ... + \beta_m e^m| > \frac{1}{h^{1+\hat{\epsilon}(h)}}, \quad h = h_1 \cdot ... \cdot h_m,$$

valid for all $\beta = (\beta_0, ..., \beta_m)^T \in \mathbb{Z}_m^m \setminus \{\overline{0}\}$, $h_i = \max\{1, |\beta_i|\}$ with

$$\hat{\epsilon}(h) = \left(4 + 7m\right)\sqrt{\log(m+1)} \over \sqrt{\log \log h}, \quad \log h \geq m^2(41 \log(m+1)+10)e^{m^2(81 \log(m+1)+20)}.$$

As far as we know, the published dependences on $m$ in $\hat{\epsilon}(h)$ have been at least quadratic and in lower bound of $\log \log h$ at least quartic. The second application of our work is presented in Leinonen’s work [8]. In a pioneer work [13] Väänänen and Zudilin proved Baker type results for a class of $q$-hypergeometric series. Following the work [13], Leinonen [8] constructed simultaneous linear forms of the type (1.4) (satisfying conditions (1.5)-(1.6) with $g(N) = N$) and proved some generalizations of the results in [13]. Moreover, in [8] Leinonen applied our Theorem 3.6 with her linear forms and gave explicit Baker type lower bounds which sharpened her results as well the results of Väänänen and Zudilin.

2. Background from metrical theory

From the general metrical theory, see [2], [3], [6], [11], [12] we get the following well known results.

**Theorem 2.1.** Let $1, \Theta_1, ..., \Theta_m \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then there exist infinitely many primitive vectors $(\beta_0, ..., \beta_m)^T \in \mathbb{Z}_m^{m+1} \setminus \{\overline{0}\}$ with $h_j := \max\{1, |\beta_j|\}, \quad j = 1, ..., m,$ satisfying

$$|\beta_0 + \beta_1 \Theta_1 + ... + \beta_m \Theta_m| < \frac{1}{\prod_{j=1}^{m} h_j}.$$ 

In the complex case Shidlovskii [12] studies linear forms over the ring of rational integers and gives the following result.

**Theorem 2.2.** [12] Let $\Theta_0 = 1, \Theta_1, ..., \Theta_m \in \mathbb{C}$ and $H \in \mathbb{Z}_{\geq 1}$ be given. Then there exists a non-zero rational integer vector $(\beta_0, \beta_1, ..., \beta_m)^T \in \mathbb{Z}_m^{m+1} \setminus \{\overline{0}\}$
with \(|\beta_j| \leq H, j = 0, 1, \ldots, m,\) satisfying

\[
|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| \leq \frac{c}{H^{(m-1)/2}}, \quad c = \sqrt{2} \sum_{j=0}^{m} |\Theta_j|.
\]

We are interested in linear forms over the ring of integers \(\mathbb{Z}_I\) in an imaginary quadratic field \(\mathbb{Q}(\sqrt{-D})\), \(D \in \mathbb{Z}_{\geq 1}, D \not\equiv 0 \pmod{4}\). For that purpose we prove

**Theorem 2.3.** Let \(\Theta_1, \ldots, \Theta_m \in \mathbb{C}\) and \(H_1, \ldots, H_m \in \mathbb{Z}_{\geq 1}\) be given. Then there exists a non-zero integer vector \((\beta_0, \beta_1, \ldots, \beta_m)^T \in \mathbb{Z}_I^{m+1} \setminus \{0\}\) with \(|\beta_j| \leq H_j, j = 1, \ldots, m,\) satisfying

\[
(2.1) \quad |\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| \leq \left(\frac{2^{7/4} D^{1/4}}{\sqrt{\pi}}\right)^{m+1} H_1 \cdots H_m,
\]

where \(\tau = 1,\) if \(D \equiv 1\) or \(2\) \((\pmod{4})\) and \(\tau = 1/2,\) if \(D \equiv 3\) \((\pmod{4})\).

3. Results

3.1. A general target. Let \(f(z)\) belong to one of the following classes of functions, enumerated by 1–3.

1. The class of Siegel’s \(G\)-functions. Typical examples are logarithm and Gauss hypergeometric functions and more generally non-entire hypergeometric series.

2. The class of Siegel’s \(E\)-functions. Typical examples are exponential and Bessel functions and more generally entire hypergeometric series.

For definition of Siegel’s \(E\)- and \(G\)-functions we refer to [6].

3. The \(q\)-hypergeometric series. Typical examples are

\[
\sum_{n=0}^{\infty} q^n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\prod_{i=1}^{n} (1 - q^i)}, \quad |q| < 1.
\]

Our Theorems 3.2, 3.4 and 3.6 are designed to be applied in the following manner. Denote \(\Theta_1 = f(\alpha_1), \ldots, \Theta_m = f(\alpha_m), \alpha_1, \ldots, \alpha_m \in \mathbb{I}^*\). Suppose that one can construct simultaneous linear forms of the type (3.2) satisfying the conditions (3.4), (3.5), (3.6) and (3.7), then our Theorem 3.2, 3.4 or 3.6 will give a Baker type lower bound for the quantity

(3.1) \(|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m|\).

It is a general phenomenon in the field of Diophantine approximations that Padé approximations and Siegel’s lemma give estimates of shape (3.6) and (3.7). However, often it is hard to find such bounds if the condition (3.5) holds, too.

3.2. A criterion. Fix now \(\Theta_1, \ldots, \Theta_m \in \mathbb{C}^*\) and write

\[
\bar{n} = (n_1, \ldots, n_m)^T, \quad N = N(\bar{n}) = n_1 + \ldots + n_m.
\]

Assume that we have a sequence of simultaneous linear forms

(3.2) \(L_{k,j}(\bar{n}) = A_{k,0}(\bar{n}) \Theta_j + A_{k,j}(\bar{n}), \quad \bar{n} \in \mathbb{Z}_I^m_{\geq 1},\)
\( k = 0, 1, \ldots, m, j = 1, \ldots, m, \) where
\[
(3.3) \quad A_{k,j} = A_{k,j}(n) \in \mathbb{Z}, \quad k, j = 0, 1, \ldots, m,
\]
satisfy a determinant condition, say,
\[
(3.4) \quad \Delta = \begin{vmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \vdots & \vdots \\ A_{m,0} & A_{m,1} & \cdots & A_{m,m} \end{vmatrix} \neq 0
\]
Further, let \( a, b, c, d, b_i, e_i \in \mathbb{R}_{\geq 0}, a > 0, \) and suppose that
\[
(3.5) \quad c, c - dm > 0,
\]
\[
(3.6) \quad |A_{k,0}(n)| \leq Q(n) = e^{q(N)},
\]
\[
(3.7) \quad |L_{k,j}(n)| \leq R_j(n) = e^{-r_j(n)},
\]
where
\[
q(N) = (aN + b \log N)g(N) + b_0N(\log N)^{1/2} + b_1N + b_2 \log N + b_3,
\]
\[-r_j(n) = (dN - cn_j)g(N) + e_0N(\log N)^{1/2} + e_1N + e_2 \log N + e_3,
\]
for all \( k, j = 0, 1, \ldots, m. \)

Let the above assumptions be valid for all \( N \geq N_l, l = 1, 2, 3 \) (where \( l \) refers to case number) in our cases:

**Case 1.**
\[
\begin{cases}
g(N) = g_1(N) := 1, \\
q(N) = q_1(N) := aN + b \log N, \\
-r_j(n) = -r_{j,1}(n) := dN - cn_j + e_2 \log N,
\end{cases}
\]
and all other \( b \)’s and \( e \)’s are zero;

**Case 2.**
\[
\begin{cases}
g(N) = g_2(N) := \log N, \quad b = 0, \\
q(N) = q_2(N) := aN \log N + b_0N(\log N)^{1/2} + b_1N + b_2 \log N + b_3, \\
-r_j(n) = -r_{j,2}(n) := (dN - cn_j) \log N + e_0N(\log N)^{1/2} + e_1N + e_2 \log N + e_3;
\end{cases}
\]

**Case 3.**
\[
\begin{cases}
g(N) = g_3(N) := N, \\
q(N) = q_3(N) := aN^2 + b_1N, \\
-r_j(n) = -r_{j,3}(n) := (dN - cn_j)N + e_1N,
\end{cases}
\]
and all other \( b \)’s and \( e \)’s are zero.

The following Theorem gives a unified result in the above three cases.

**Theorem 3.1.** Under the above assumptions there exist explicit positive constants \( F_l \) and \( G_l \) not depending on \( H \) such that
\[
(3.8) \quad |\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| > F_l \left( \prod_{j=1}^{m} (2mH_j) \right)^{-\frac{-a}{2m} - e_l(H)}
\]
for all \( \beta = (\beta_0, \beta_1, ..., \beta_m)^T \in \mathbb{Z}_t^{m+1} \setminus \{0\} \) and
\[
H = \prod_{j=1}^{m} (2mH_j) \geq G_l, \quad H_j \geq h_j = \max\{1, |\beta_j|\},
\]
with an error term \( \epsilon_l(H) \to 0 \).

In subsections 3.3-3.5 we consider the three cases more closely.

3.3. Case 1.

**Theorem 3.2.** Denote \( f = \frac{2}{c - dm} \) and
\[
A_1 = \frac{acm}{c - dm} + B_1 \log(ef), \quad B_1 = \frac{ae_2m}{c - dm} + b.
\]
Then
\[
F_1^{-1} = 2e^{A_1}, \quad \epsilon_1(H) = B_1 \frac{\log \log H}{\log H}
\]
and
\[
G_1 = \max\{m, N_1, e^{x_1/f}\}, \quad x_1 = \max\{S_1, 1\},
\]
where \( S_1 \) is the largest solution of the equation
\[
S = f(e^2m \log S + dm^2 + e^{2m}).
\]

3.4. Case 2. Before stating our results we introduce a function \( z : \mathbb{R} \to \mathbb{R} \), the inverse function of the function \( y(z) = z \log z, \quad z \geq 1/e \), considered in [7].

**Lemma 3.3.** [7] The inverse function \( z(y) \) of the function \( y(z) = z \log z, \quad z \geq 1/e \), is strictly increasing. Define \( z_0(y) = y \) and \( z_n(y) = \frac{y}{\log \log \log \cdots} \) for \( n \in \mathbb{Z}^+ \). Suppose \( y > e \), then \( z_1 < z_3 < \cdots < z < \cdots < z_2 < z_0 \). Thus the inverse function may be given by the infinite nested logarithm fraction
\[
z(y) = \lim_{n \to \infty} z_n(y) = \frac{y}{\log \frac{y}{\log \log \cdots}}
\]
for \( y > e \). In particular,
\[
z(y) < z_2(y) = \frac{y}{\log \frac{y}{\log y}}
\]
for \( y > e \).

**Theorem 3.4.** Denote now \( f = \frac{2}{c - dm} \) and
\[
A_2 = b_0 + \frac{ae_0m}{c - dm}, \quad B_2 = a + b_0 + b_1 + \frac{ae_1m}{c - dm},
\]
\[
C_2 = am + b_2 + \frac{a(dm^2 + e^{2m})}{c - dm}, \quad D_2 = b_0m + \frac{ae_0m^2}{c - dm},
\]
\[
E_2 = (a + b_0 + b_1)m + b_2 + b_3 + \frac{a((2d + 2e_0 + e_3)m^2 + (e_2 + e_3)m)}{c - dm}.
\]
Then
\[
F_2^{-1} = 2e^{E_2}
\]
and
\[
\epsilon_2(H) = \xi(z, H) :=
\]
with

\[ G_2 = \max \{ m, N_2, e^{(x_2 \log x_2)/f}, e^{c/f} \}, \quad x_2 = \max \{ S_2, 1 \}, \]

where \( S_2 \) is the largest solution of the equation

\[ S \log S = f(e_0 m S)(\log S)^{1/2} + e_1 m S + (d m^2 + e_2 m) \log S 
   + e_0 m^2 (\log S)^{1/2} + 2 d m^2 + 2 e_0 m^2 + e_1 m^2 + e_2 m + e_3 m. \]

In this case the estimate corresponding to (3.8) may be written as follows

\[ |\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| \geq \]

\[ F_2 (z(f \log H))^{-C_2} H^{-\frac{m}{e-dm} - A_2 (f \log H)^{1/2} - B_2 \log (f \log H) - D_2 (\log (f \log H))^{1/2}}. \]

Note, that

\[ z(f \log H) < z_2(f \log H) \]

for \( f \log H > e \) by (3.12) and thus

\[ e_2(H) = \xi(z, H) < \xi(z_2, H) \]

for \( f \log H > e \). Write now

\[ \rho_2(x) = \frac{\log x}{\log x - \log \log x}. \]

Then (3.17) may further be estimated by using

\[ z_2(f \log H) \leq \rho_2(x_0) f \left( 1 - \frac{\log f}{\log(f \log H)} \right) \frac{\log H}{\log \log H} \]

valid for all

\[ f \log H \geq x_0 \geq e^e, \quad H > e. \]

Note, that if \( 0 < c - d m \leq 2 \), then

\[ z_2(f \log H) \leq \rho_2(x_0) f \frac{\log H}{\log \log H}. \]

By using the estimate (3.21) we get the following corollary where the lower bound in (3.23) is a generalization of what we see in the works on E-functions.

**Corollary 3.5.** Write \( \rho = \rho_2(x_0) \). If \( 0 < c - d m \leq 2 \), \( H > e \) and

\[ f \log H \geq x_0 := \max \{ f \log m, f \log N_2, x_2 \log x_2, e^e \}, \]

then

\[ |\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| \geq \]

\[ \frac{1}{2 e^{E_2(\log H)^2}} \left( \frac{\log H}{\log H} \right)^{C_2} H^{-\frac{m}{e-dm} - A_2 \log \log H - B_2 \log \log H - D_2 \log \log H.} \]
In [4], \(c = 1, d = 0\), so Corollary 3.5 applies.

In most of the existing works only the terms corresponding to \(A_2\) and \(C_2\) are presented and usually only a main term is given including the other terms implicitly. Hence in such a situation explicit dependence on the parameters, say for example on \(m\), may become invisible. Next we like to mention that all the methods applied to \(E\)-functions seem to yield the situation where \(A_2 \neq 0\). If we had \(A_2 = 0\), then the terms with \(B_2\) and \(C_2\) would become more important. That would be the case if e.g. one could find appropriate explicit Padé type approximations instead of those produced by Siegel’s lemma.

3.5. Case 3.

**Theorem 3.6.** Now we have

\[
F^{-1}_3 = 2e^{B_3}, \quad \epsilon_3(H) = A_3 \frac{1}{\sqrt{\log H}}, \quad G_3 = \max\{m, N_3, e\},
\]

where general \(A_3\) and \(B_3\) are given in the proof section. In the particular case, \(b_1 = e_1 = 0\), they read

\[
A_3 = \frac{2acm}{(c - dm)^{3/2}}, \quad B_3 = \frac{acm^2(c + dm + 2\sqrt{cdm})}{(c - dm)^2}.
\]

4. Proofs

4.1. **Proof of Theorem 2.3.** For \(D \in \mathbb{Z}_{\geq 1}, D \not\equiv 0 \pmod{4}\) the ring of integers may be given by \(\mathbb{Z}_I = \mathbb{Z} + \mathbb{Z}(h + l\sqrt{-D})\) with \(h = 0, l = 1\), if \(D \equiv 1\) or \(2 \pmod{4}\) and \(h = l = 1/2\), if \(D \equiv 3 \pmod{4}\).

We start with a simple principle. First we define a lattice

\[
\lambda = \mathbb{Z}(1, 0) + \mathbb{Z}(h, l\sqrt{-D}), \quad \det \lambda = \sqrt{-D}2^{-2h}
\]

and a complex disk

\[
D_R = \{x + y(h + l\sqrt{-D}) \in \mathbb{C} \mid x, y \in \mathbb{R}, \ |x + y(h + l\sqrt{-D})| \leq R\}.
\]

with a radius \(R > 0\) and a corresponding real disk

\[
C_R = \{(v, w)^T \in \mathbb{R}^2 \mid v^2 + w^2 \leq R^2\}, \quad \text{Vol } C_R = \pi R^2.
\]

Then

\[
(4.1) \quad x + y(h + l\sqrt{-D}) \in D_R \cap \mathbb{Z}_I \iff (x + yh, yl\sqrt{-D})^T \in C_R \cap \lambda.
\]

Next we define a lattice

\[
(4.2) \quad \Lambda = \mathbb{Z}\vec{l}_1 + ... + \mathbb{Z}\vec{l}_{2m+2} \subseteq \mathbb{R}^{2m+2}
\]

generated by

\[
\begin{align*}
\vec{l}_1 &= (1, 0, 0, 0, ..., 0, 0)^T, \quad \vec{l}_2 = (h, l\sqrt{-D}, 0, 0, ..., 0, 0)^T, \\
\vec{l}_3 &= (0, 0, 1, 0, ..., 0, 0)^T, \quad \vec{l}_4 = (0, 0, h, l\sqrt{-D}, 0, 0, ..., 0, 0)^T, \\
... \\
\vec{l}_{2m+1} &= (0, 0, ..., 0, 0, 1, 0)^T, \quad \vec{l}_{2m+2} = (0, 0, ..., 0, h, l\sqrt{-D})^T.
\end{align*}
\]
Immediately, \( \det \Lambda = \left( \sqrt{D} 2^{-2h} \right)^{m+1} \).

By using the following notations
\[
a + b(h + l\sqrt{-D}) = -(z_1\Theta_1 + \ldots + z_m\Theta_m), \quad z_k = x_k + y_k(h + l\sqrt{-D}),
\]
\[
v_k = x_k + y_k h, \quad w_k = y_k l\sqrt{D}, \quad x_k, y_k \in \mathbb{R}, \quad k = 0,1,\ldots,m,
\]
we define the following sets
\[
D = \{ (z_0, z_1, \ldots, z_m)^T \in \mathbb{C}^{m+1} \mid |z_0 - (a + b(h + l\sqrt{-D}))| \leq R_0; |z_k| \leq H_k, k = 1, \ldots, m\},
\]
\[
C = \{ (v_0, w_0, v_1, w_1, \ldots, v_m, w_m)^T \in \mathbb{R}^{2m+2} \mid (v_0 - (a + bh))^2 + (w_0 - bl\sqrt{D})^2 \leq R_0^2, v_k^2 + w_k^2 \leq H_k^2, k = 1, \ldots, m\}.
\]
First we note that \( C \) is a symmetric convex body. For the volume of \( C \) we get
\[
\text{Vol} \ C = \int \ldots \int \left( \int \int \int \int \right) dv_0 dw_0 \cdot dv_1 dw_1 \cdot \ldots \cdot dv_m dw_m = \pi R_0^2 \int \ldots \int \left( \int \int \int \right) dv_1 dw_1 dw_2 \cdot \ldots \cdot dw_m = \ldots = \pi^{m+1} H_1^2 \cdots H_m^2 R_0^2 = \pi^{m+1} H_1^2 \cdots H_m^2 \left( \frac{2^{2h} \sqrt{D}}{\pi} \right)^{m+1} \frac{1}{H_1^2 \cdots H_m^2} = 2^{2m+2} \left( \frac{\sqrt{D}}{2^{2h}} \right)^{m+1} = 2^{2m+2} \det \Lambda.
\]
Thus by Minkowski’s convex body theorem, see [11], there exists a non-zero lattice vector
\[
(x_0 + y_0 h, y_0 l\sqrt{D}, \ldots, x_m + y_m h, y_m l\sqrt{D})^T \in C \cap \Lambda \setminus \{0\}.
\]
Consequently, by the above principle (4.1), we get a non-zero integer vector
\[
(\beta_0, \beta_1, \ldots, \beta_m)^T = (x_0 + y_0 (h + l\sqrt{-D}), \ldots, x_m + y_m (h + l\sqrt{-D}))^T \in D \cap \mathbb{Z}_l^{m+1} \setminus \{0\}
\]
with \(|\beta_k| \leq H_k, k = 1, \ldots, m, \) satisfying
\[
|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| \leq \left( \frac{2^h D^{1/4}}{\sqrt{\pi}} \right)^{m+1} \frac{1}{H_1 \cdots H_m}. \quad \Box
\]

4.2. **Proof of Theorems 3.1–3.6.** Our proof starts in a classical manner and after that we give a rough description how to get Baker type estimates. Next we will introduce our tuning process which allows us to continue from the classical startup.
4.2.1. A classical start. We use the notation

\[ \Lambda := \beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m, \quad \beta_j \in \mathbb{Z} \]

for the linear form to be estimated. Using our simultaneous linear forms

\[ L_{k,j}(\overline{n}) = A_{k,0}(\overline{n}) \Theta_j + A_{k,j}(\overline{n}) \]

from (3.2) we get

\[ A_{k,0} \Lambda = \Omega_k + \beta_1 L_{k,1}(\overline{n}) + \ldots + \beta_m L_{k,m}(\overline{n}) \]

where

\[ \Omega_k = \Omega_k(\overline{n}) = A_{k,0}(\overline{n}) \beta_0 - \beta_1 A_{k,1}(\overline{n}) - \ldots - \beta_m A_{k,m}(\overline{n}) \in \mathbb{Z}^d. \]

If now \( \Omega_k \neq 0 \), then by (3.6), (3.7), (3.9), (4.5) and (4.6) we get

\[ 1 \leq |\Omega_k| = |A_{k,0} \Lambda - (\beta_1 L_{k,1} + \ldots + \beta_m L_{k,m})| \leq \]

\[ |A_{k,0}| |\Lambda| + \sum_{j=1}^{m} |\beta_j| |L_{k,j}| \leq Q(\overline{n}) |\Lambda| + \sum_{j=1}^{m} H_j R_j(\overline{n}). \]

Here we want to have, say

\[ \sum_{j=1}^{m} H_j R_j(\overline{n}) \leq \frac{1}{2}, \]

in order to get a lower bound

\[ 1 \leq 2|\Lambda| Q(\overline{n}) \]

for our linear form \( \Lambda \).

4.2.2. A rough version. Here we outline a rough version of the proof by studying the case \( b = b_0 = b_1 = b_2 = b_3 = e_0 = e_1 = e_2 = e_3 = 0 \), for simplicity. It starts by fixing the remainders and heights:

\[ H_j R_j(\overline{n}) = \frac{1}{2m} \iff 2m H_j = e^{r_j(\overline{n})} = e^{(-dN + cn)} g(N) \Rightarrow \]

\[ e^{(-dN + e \sum_{j=1}^{m} n_j) g(N)} = e^{(c - dm) N g(N)} = \prod_{j=1}^{m} (2m H_j) \Rightarrow \]

\[ Q(\overline{n}) = e^{a N g(N)} = \left( \prod_{j=1}^{m} (2m H_j) \right)^{-\frac{a}{c - dm}} \Rightarrow \]

\[ 1 \leq 2|\Lambda| Q(\overline{n}) = 2|\Lambda| \left( \prod_{j=1}^{m} (2m H_j) \right)^{-\frac{a}{c - dm}} . \]
4.2.3. Tuning. Now a direct generalization of the second equality of (4.10) would be

\[ r_j(\bar{n}) = \log(2mH_j), \]

where

\[ r_j(\bar{n}) = (-dN + cn_j)g(N) - e_0N(\log N)^{1/2} - e_1N - e_2\log N - e_3. \]

However, (4.11) will be too rough and thus we tune it into right frequency by defining

\[ B_j = \log(2mH_j) + dm\hat{g}_l(W) + e_0m((\log W)^{1/2} + 2) + e_1m + e_2, \]

where

\[ \hat{g}_1(W) = 1, \quad \hat{g}_2(W) = \log W + 2, \quad \hat{g}_3(W) = 2W + m, \]

corresponding to our three cases. Now we state a new system of equations

\[ \sum_{j=1}^{m} w_j = W, \]
\[ r_j(\bar{w}) = B_j, \quad j = 1, \ldots, m. \]

Here (4.14) reads

\[ \log(2mH_j) + dm\hat{g}_l(W) + e_0m((\log W)^{1/2} + 2) + e_1m + e_2 \]

which by (4.13) gives

\[ \sum_{j=1}^{m} w_j = W, \]
\[ r_j(\bar{w}) = B_j, \quad j = 1, \ldots, m. \]

The equation (4.16) has a solution \( W \geq m \), if \( H \) is big enough. Then we choose the largest, say \( S := W_L \geq m \). (Any solution \( W \geq 1 \) would be satisfactory but for technical reasons we choose \( W \geq m \).) From our assumptions it follows that \( m \geq 2, c > 0, g(S) \geq 1, g_l(S) \geq 1 \) for \( l = 1, 2, 3 \), and \( H_j \geq 1 \) for \( j = 1, \ldots, m \). Hence \( B_j \geq \log 4 \) for \( j = 1, \ldots, m \), which by (4.15) implies

\[ s_j := w_j = \frac{B_j + e_0S((\log S)^{1/2} + e_1S + e_2\log S + e_3 + dSg(S))}{cg(S)} > \frac{\log 4}{cg(S)} > 0. \]

Consequently, also the estimate (4.17) is valid for \( H \) big enough (independently of each individual term \( H_j \)).

Put \( \sigma_j = [s_j] \) and write \( \bar{\sigma} = (\sigma_1, \ldots, \sigma_m)^T, \bar{1} = (1, \ldots, 1)^T \), then

\[ \sigma \leq \bar{\sigma} < \bar{\sigma} + \bar{1}. \]

First we note that

\[ T := N(\bar{\sigma} + \bar{1}) = N(\bar{\sigma}) + m \leq N(\bar{\sigma}) + m = S + m, \quad S < T. \]
Next we give an estimate for the difference
\begin{equation}
\begin{aligned}
r_j(\sigma) - r_j(\sigma + \bar{T}) = \\
\left(-dN(\sigma) + cs_jg(S) - e_0N(\sigma)(\log N(\sigma))^{1/2} - e_1N(\sigma) - e_2\log N(\sigma) - e_3 \\
- ((-dN(\sigma + \bar{T}) + c(\sigma_j + 1))g(N(\sigma + \bar{T})) - e_0N(\sigma + \bar{T})(\log N(\sigma + \bar{T}))^{1/2} \\
- e_1N(\sigma + \bar{T}) - e_2\log N(\sigma + \bar{T}) - e_3 = \\
d(Tg(T) - Sg(S)) + c(s_jg(S) - (\sigma_j + 1)g(T)) \\
+ e_0(T(\log T)^{1/2} - S(\log S)^{1/2}) + e_1(T - S) + e_2(\log T - \log S).
\end{aligned}
\end{equation}

By \( s_j < \sigma_j + 1 \), the increasing property of \( g(x) \) and the mean value theorem we get
\begin{equation}
r_j(\sigma) - r_j(\sigma + \bar{T}) \leq \\
d(Tg(T) - Sg(S)) + e_0m((\log S)^{1/2} + 2) + e_1m + e_2.
\end{equation}

Hence
\begin{equation}
r_j(\sigma) < r_j(\sigma + \bar{T}) + dm\hat{g}(S) + e_0m((\log S)^{1/2} + 2) + e_1m + e_2, \quad l \in \{1, 2, 3\},
\end{equation}
which is the reason to define (4.12).

According to the non-vanishing of the determinant (3.4) and the assumption \( \vec{\beta} = (\beta_0, \beta_1, ..., \beta_m)^T \neq \vec{0} \) it follows that
\begin{equation}
\Omega_k(\sigma + \bar{T}) \in \mathbb{Z}_t \setminus \{0\}
\end{equation}
with some integer \( k \in [0, m] \). Now we are ready to prove the essential estimate
\begin{equation}
\sum_{j=1}^{m} H_j R_j(\sigma + \bar{T}) \leq \sum_{j=1}^{m} H_j e^{-r_j(\sigma + \bar{T})} \quad (4.22)
\end{equation}
\begin{equation}
\sum_{j=1}^{m} H_j e^{-H_j + dm\hat{g}(S) + e_0m((\log S)^{1/2} + 2) + e_1m + e_2} = \frac{1}{2}.
\end{equation}

Hence by (4.7) we get
\begin{equation}
1 < 2|\Lambda|Q(\sigma + \bar{T}) = 2|\Lambda|e^{q(N(\sigma + \bar{T}))} \leq 2|\Lambda|e^{q(S+m)},
\end{equation}
where
\begin{equation}
q(S + m) = (a(S + m) + b\log(S + m))g(S + m) + b_0(S + m)(\log(S + m))^{1/2} + b_1(S + m) + b_2\log(S + m) + b_3.
\end{equation}

Because \( g(x) \) is increasing we get
\begin{equation}
g(S + m) = g(S) + mV(S), \quad V(S) = \max_{S \leq x \leq S + m} \{g'(x)\}.
\end{equation}

Or, remembering the assumption \( m \leq S \), we may use the following estimates
\begin{equation}
\log(S + m) \leq \log S + 1, \quad (\log(S + m))^{1/2} \leq (\log S)^{1/2} + 1.
\end{equation}
Consequently
\begin{equation}
q(S + m) \leq aSg(S) + Y(S),
\end{equation}
where

\[ Y(S) = amg(S) + amSV(S) + am^2V(S) + bg(S + m)\log(S + m) + \\
\quad b_0(S + m)(\log(S + m))^{1/2} + b_1(S + m) + b_2\log(S + m) + b_3. \]

From (4.16) we get

\[ (4.29) \quad Sg(S) = \frac{\log H}{c - dm} + \frac{X(S)}{c - dm}, \]

where

\[ X(S) = dm^2\hat{g}(S) + e_0mS(\log S)^{1/2} + e_1mS + e_2m \log S + \\
\quad e_0m^2((\log S)^{1/2} + 2) + e_1m^2 + e_2m + e_3m. \]

Hence

\[ (4.30) \quad Q(\sigma + 1) \leq H^{\frac{a}{c - dm} + Z(S)}, \quad Z(S) = \frac{1}{\log H} \left( \frac{a}{c - dm}X(S) + Y(S) \right). \]

In the following we will consider \( S \) as a variable greater than \( W_L \).

4.2.4. Case 1. We have \( \hat{g}(S) = 1 \) and thus

\[ Z(S) = \frac{1}{\log H} \left( \frac{a}{c - dm}(dm^2 + e_2m \log S + e_2m) + am + b \log(S + m) \right) \leq \\
\quad \frac{1}{\log H} \left( \frac{a(dm^2 + e_2m)}{c - dm} + am + b \right) + \frac{\log S}{\log H} \left( \frac{ae_2m}{c - dm} + b \right). \]

Here (4.16) reads

\[ (4.31) \quad (c - dm)W - dm^2 - e_2m \log W - e_2m = \log H. \]

Let \( W_1 \) denote the largest solution of the equation

\[ (c - dm)W - dm^2 - e_2m \log W - e_2m = \frac{1}{2}(c - dm)W. \]

Hence

\[ (4.32) \quad (c - dm)S - dm^2 - e_2m \log S - e_2m \geq \frac{1}{2}(c - dm)W_1, \]

holds for all \( S \geq x_1 := \max\{W_1, W_L, m\} \). Further, we choose \( H \) such that

\[ (4.33) \quad x_1 \leq S \leq f \log H, \quad f = \frac{2}{c - dm}. \]

Thus

\[ (4.34) \quad Z(S) \leq A_1 \frac{1}{\log H} + B_1 \frac{\log \log H}{\log H}, \]

where

\[ B_1 = \frac{ae_2m}{c - dm} + b, \]

\[ A_1 = \frac{adm^2}{c - dm} + am + \frac{ae_2m}{c - dm} + b + B_1 \log f = \frac{acm}{c - dm} + B_1 \log(ef). \]
Hence

\[ 1 < 2|\Lambda|Q(\pi + \Gamma) \leq |\Lambda|2e^{A_1} H \frac{\log H}{c - dm} + B_1 \frac{\log \log H}{\log H} , \]

where \( \Lambda = \beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m \) is our linear form. This proves Theorem 3.2.

4.2.5. Case 2. Here

\[ q_2(S + m) \leq a(S + m) \log(S + m) + b_0(S + m)(\log(S + m))^{1/2} + \]

\[ b_1(S + m) + b_2 \log(S + m) + b_3 \leq aS \log(S) + Y(S), \]

\[ Y(S) = b_0S(\log S)^{1/2} + (a + b_0 + b_1)S + (am + b_2) \log S + b_0m(\log S)^{1/2} + (a + b_0 + b_1)m + b_2 + b_3. \]

From (4.16) we get

\[ S \log S = \frac{\log H}{c - dm} + \frac{X(S)}{c - dm}, \]

where

\[ X(S) = dm^2 g_2(S) + e_0mS(\log S)^{1/2} + e_1mS + e_2m \log S + e_0m^2(\log S)^{1/2} + (2e_0 + e_1)m^2 + e_2m + e_3m, \quad g_2(S) = \log S + 2. \]

Hence, by (4.30),

\[ Z(S) = \frac{1}{\log H} \left( A_2 S(\log S)^{1/2} + B_2 S + C_2 \log S + D_2(\log S)^{1/2} + E \right), \]

where

\[ A_2 = b_0 + \frac{ae_0m}{c - dm}, \quad B_2 = a + b_0 + b_1 + \frac{ae_1m}{c - dm}, \]

\[ C_2 = am + b_2 + \frac{a(dm^2 + e_2m)}{c - dm}, \quad D = b_0m + \frac{ae_0m^2}{c - dm}, \]

\[ E_2 = (a + b_0 + b_1)m + b_2 + b_3 + \frac{a((2d + 2e_0 + e_1)m^2 + (e_2 + e_3)m)}{c - dm}. \]

Here (4.16) has the form

\[ (c - dm)W \log W - dm^2(\log W + 2) - e_0mW(\log W)^{1/2} - e_1mW - e_2m \log W - e_0m^2((\log W)^{1/2} + 2) - e_1m^2 - e_2m - e_3m = \log H. \]

Let \( W_2 \) denote the largest solution of the equation

\[ (c - dm)W \log W - dm^2(\log W + 2) - e_0mW(\log W)^{1/2} - e_1mW - e_2m \log W - e_0m^2((\log W)^{1/2} + 2) - e_1m^2 - e_2m - e_3m = \frac{c - dm}{2} W \log W. \]

Assume then \( S \geq x_2 := \max\{W_2, W, m\} \). Analogously to Case 1 we may choose \( H \) such that

\[ S \log S \leq f \log H, \quad f = \frac{2}{c - dm}. \]
By (3.12) we get
\begin{equation}
S \leq z(f \log H) \leq z_2(f \log H) = \frac{f \log H}{\log \frac{f \log H}{\log(f \log H)}}
\end{equation}
valid for
\begin{equation}
f \log H > \varepsilon.
\end{equation}
Note the estimate
\begin{equation}
S \left( \frac{\log S}{\log H} \right) \frac{1}{2} \log \frac{\log H}{\log f \log H} = S \left( \frac{\log S}{\log H} \right) \frac{1}{2} \log \frac{\log H}{\log f \log H} \leq \left( f \log H \right)^{1/2}
\end{equation}
valid for
\begin{equation}
(4.42)
f \log H > \varepsilon.
\end{equation}
Note that
\begin{equation}
B_2 \frac{z(f \log H)}{\log H} = o \left( A_2 \left( f \frac{z(f \log H)}{\log H} \right) \frac{1}{2} \right)
\end{equation}
and similarly to the terms involving \( C_2 \) and \( D_2 \). Thus
\begin{equation}
A_2 \left( f \frac{z(f \log H)}{\log H} \right) \frac{1}{2}
\end{equation}
will be the main error term, for any \( H \) big enough, if \( A_2 \neq 0 \).

Further, we note that the estimate (4.43) may be written as follows
\begin{equation}
Q(\sigma + \Gamma) \leq H \left( \frac{z(f \log H)}{\log H} \right) + \xi(z, H),
\end{equation}
where the error term satisfies
\begin{equation}
\xi(z, H) \leq \xi(z_2, H).
\end{equation}
Next we shall prove the estimates (3.19), (3.21) under the assumption (3.20).

First we get
\begin{equation}
z_2(y) = \frac{y}{\log y - \log \log y} \leq \frac{\log x_0}{\log x_0 - \log \log x_0} \frac{y}{\log y} = \rho_2(x_0) \frac{y}{\log y},
\end{equation}
to be valid for all \( y \geq x_0 \geq e^e \). Further, we have
\begin{equation}
z_2(fy) \leq \rho_2(x_0) f \frac{y}{\log fy} = \rho_2(x_0) f \left( 1 - \frac{\log f}{\log fy} \right) \frac{y}{\log y}.
\end{equation}
for $fy \geq x_0$. In particular, we have
\[(4.47)\]
\[z_2(f \log H) \leq \rho_2(x_0)f \left(1 - \frac{\log f}{\log(f \log H)}\right) \leq \rho_2(x_0)f \frac{\log H}{\log \log H}\]
for all
\[f \log H \geq x_0 \geq e^e, \quad H > e,
\]
where the last inequality in (4.47) is valid with $0 < c - dm \leq 2$. Hence
\[(4.48)\]
\[Q(\sigma + T) \leq e^{E_2} \left(f \rho \frac{\log H}{\log \log H}\right) \leq 2 \sqrt{e} \log H \left(\frac{a_3}{\log \log H} + A_3 \sqrt{e} \log \log H\right),
\]
if $\rho \geq \rho_2(x_0)$, by (4.44), (4.47). Now substitute (4.43), (4.45) and (4.48), respectively, into
\[(4.49)\]
\[1 < 2|\Lambda|Q(\sigma + T),
\]
proving (3.13), (3.16) and (3.23). This ends the proof of Theorem 3.4 and Corollary 3.5.

4.2.6. Case 3. Here $\hat{g}_3(S) = 2S + m$, so (4.16) reads
\[(4.50)\]
\[(c - dm)W^2 - (2dm^2 + e_1 m)W - dm^3 - e_1 m^2 = \log H.
\]
Now we simply choose the larger solution
\[(4.51)\]
\[S = \frac{2dm^2 + e_1 m + \sqrt{(2dm^2 + e_1 m)^2 + 4(dm^3 + e_1 m^2 + \log H)(c - dm)}}{2(c - dm)}.
\]
For convenience, we will use the following estimate
\[(4.52)\]
\[1 = S_3 \leq S \leq v_1 + v_2 \sqrt{\log H}, \quad v_2 = \frac{1}{\sqrt{c - dm}},
\]
\[v_1 = \frac{2dm^2 + e_1 m + \sqrt{e_1^2 m^2 + 4cdm^3 + 4ce_1 m^2}}{2(c - dm)}.
\]
Now, by using (4.50) and (4.52), we get
\[q_3(S + m) = a(S + m)^2 + b_1 (S + m) =
\]
\[\frac{a}{c - dm} \log H + \left(\frac{a(2dm^2 + e_1 m)}{c - dm} + 2am + b_1\right) S + \frac{a(dm^3 + e_1 m^2)}{c - dm} + am^2 + b_1 m \leq
\]
\[\frac{a}{c - dm} \log H + v_2 w_1 \sqrt{\log H} + v_1 w_1 + w_2,
\]
where
\[w_1 = \frac{a(2dm^2 + e_1 m)}{c - dm} + 2am + b_1, \quad w_2 = \frac{a(dm^3 + e_1 m^2)}{c - dm} + am^2 + b_1 m.
\]
Hence
\[Q(\sigma + T) \leq H \frac{a}{c - dm} + \frac{b_3}{\log H} + A_3 \frac{1}{\log \log H} = e^{B_3} H \frac{a}{c - dm} + A_3 \frac{1}{\log \log H},
\]
where
\[A_3 = v_2 w_1, \quad B_3 = v_1 w_1 + w_2.\]
In particular, if $b_1 = e_1 = 0$, then
\[ A_3 = \frac{2acm}{(c - dm)^{3/2}}, \quad B_3 = \frac{acm^2(c + dm + 2\sqrt{cdm})}{(c - dm)^2}. \]
This proves Theorem 3.6.

4.2.7. The term $G_l$. Yet we need to determine terms $G_l$, $l = 1, 2, 3$. In each case, there are some assumptions imposed on $H$. The determinant condition (3.4) and the conditions $S \geq m$, (4.34) and (4.40) should be satisfied. So, if we put $f_1 = x_1/f$, $f_2 = (x_2 \log x_2)/f$, $f_3 = S_3$ and suppose
\begin{equation}
H \geq G_l := \max\{m, N_l, e^h\},\end{equation}
then Theorem 3.1 is proved. Finally we note, that in Corollary 3.5 we need the assumption (3.20), too. The condition (4.53) applied in the Case 2 shows, in particular, that
\begin{equation}
f \log H \geq f \log G_2 \geq x_2 \log x_2\end{equation}
and thus in (3.22) we may choose
\[ \rho = \frac{\log(x_0)}{\log(x_0) - \log \log(x_0)}, \quad x_0 = \max\{f \log m, f \log N_2, x_2 \log x_2, e^e\}. \] □

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References


