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UPPER CONICAL DENSITY RESULTS FOR GENERAL MEASURES ON $\mathbb{R}^n$

MARIANNA CSÖRNYEI[1], ANTTI KÄENMÄKI[2], TAPIO RAJALA[2]
AND VILLE SUOMALA[2]

[1]*Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK* (mari@math.ucl.ac.uk)
[2]*Department of Mathematics and Statistics, PO Box 35 (MaD), 40014 University of Jyväskylä, Finland*
(antti.kaenmaki@jyu.fi; tapio.m.rajala@jyu.fi; ville.suomala@jyu.fi)

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Dedicated to Professor Pertti Mattila on the occasion of his 60th birthday

Abstract We study conical density properties of general Borel measures on Euclidean spaces. Our results are analogous to the previously known result on the upper density properties of Hausdorff and packing-type measures.

Keywords: upper conical density; Hausdorff dimension; homogeneity of measures; rectifiability

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1. Introduction

The extensive study of upper conical density properties for Hausdorff measures was pioneered by Besicovitch, who studied the conical density properties of purely 1-unrectifiable fractals on the plane. Since Besicovitch's time upper density results have played an important role in geometric measure theory. Due to the works of Marstrand [**7**], Salli [**12**], Mattila [**9**], and others, the upper conical density properties of Hausdorff measures $\mathcal{H}^s$ for all values of $0 \leqslant s \leqslant n$ are very well understood. There are also analogous results for many (generalized) Hausdorff and packing measures (see [**4**] and references therein). Conical density results are useful since they give information on the distribution of the measure if the values of the measure are known on some small balls. The main applications deal with rectifiability [**10**], but often upper conical density theorems may also be viewed as some kind of anti-porosity theorems (see [**9**] and [**4**] for more on this topic).

When working with a Hausdorff or packing-type measure μ, it is useful to study densities such as

$$\limsup_{r\downarrow 0} \mu(X(x,r,V,\alpha))/h(2r),$$

where h is the gauge function used to construct the measure μ and $X(x,r,V,\alpha)$ is a cone around the point x (see § 2 for the formal definition). However, most measures are so unevenly distributed that there are no gauge functions that could be used to approximate the measure in small balls. This is certainly the case for many self-similar and multifractal-type measures. For these measures the above quoted results give no information. To obtain conical density results for general measures it seems natural to replace the value of the gauge h in the denominator by the measure of the ball $B(x,r)$ and consider upper densities such as

$$\limsup_{r\downarrow 0} \mu(X(x,r,V,\alpha))/\mu(B(x,r)).$$

Our purpose in this paper is to study densities of this type, and more general types, for locally finite Borel regular measures on $\mathbb{R}^n$. In particular, we will answer some of the problems posed in [**4**].

The paper is organized as follows. In § 2, we set up some notation and discuss auxiliary results that will be needed later on. In particular, we recall a dimension estimate for average homogeneous measures obtained in [**3**]. In § 3, we prove an upper density result valid for all locally finite Borel regular measures on $\mathbb{R}^n$. The result gives a positive answer to [**4**, Question 4.3]. It shows that around typical points a locally finite Borel regular measure cannot be distributed, so that it lies mostly on only one one-sided cone at all small scales. In § 4, we obtain more detailed information on the distribution of the measure μ provided that its Hausdorff dimension is bounded from below. The result, Theorem 4.1, is analogous to the results of [**4, 5, 9**], obtained before for Hausdorff and packing-type measures, and it gives strong insight into [**4**, Question 4.1]. In § 5, we give a negative answer to [**4**, Question 4.2] and, moreover, we show that Theorem 4.1 is not valid if we only assume that the measure is purely m-unrectifiable.

2. Notation and preliminaries

We start by introducing some notation. Let $n \in \mathbb{N}$, $m \in \{0,\dots,n-1\}$, and $G(n,n-m)$ denote the space of all $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$. The unit sphere of $\mathbb{R}^n$ is denoted by S^{n-1}. For $x \in \mathbb{R}^n$, $\theta \in S^{n-1}$, $0 \leqslant \alpha \leqslant 1$ and $V \in G(n,n-m)$, we set

$$\begin{aligned} H(x,\theta,\alpha) &= \{y \in \mathbb{R}^n : (y-x)\cdot\theta > \alpha|y-x|\},\\ X^+(x,\theta,\alpha) &= H(x,\theta,(1-\alpha^2)^{1/2}),\\ X(x,V,\alpha) &= \{y \in \mathbb{R}^n : \operatorname{dist}(y-x,V) < \alpha|y-x|\}. \end{aligned}$$

We also denote $X^+(x,r,\theta,\alpha) = B(x,r) \cap X^+(x,\theta,\alpha)$ and $X(x,r,V,\alpha) = B(x,r) \cap X(x,V,\alpha)$, where $B(x,r)$ is the closed ball centred at x with radius $r > 0$. Observe that $X^+(x,\theta,\alpha)$ is the one side of the two-sided cone $X(x,\ell,\alpha)$, where $\ell \in G(n,1)$ is the line pointing to the direction θ. We usually use the 'X notation' for very narrow cones, whereas the 'H cones' are considered as 'almost half-spaces'. If $V \in G(n,n-m)$, we denote the orthogonal projection onto V by proj_V. Furthermore, if $B = B(x,r)$ and $t > 0$, then by the notation tB, we mean the ball $B(x,tr)$.

By a measure we will always mean a finite non-trivial Borel regular (outer) measure defined on all subsets of some Euclidean space $\mathbb{R}^n$. Since all our results are local, and valid only almost everywhere, we could easily replace the finiteness condition by assuming that μ is almost everywhere locally finite in the sense that $\mu(\{x \in \mathbb{R}^n : \mu(B(x,r)) = \infty \text{ for all } r > 0\}) = 0$. The support of the measure μ is denoted by $\mathrm{spt}(\mu)$. The *(lower) Hausdorff dimension of the measure* μ is defined by

$$\dim_{\mathrm{H}}(\mu) = \inf\{\dim_{\mathrm{H}}(A) : A \text{ is a Borel set with } \mu(A) > 0\},$$

where $\dim_{\mathrm{H}}(A)$ denotes the Hausdorff dimension of the set $A \subset \mathbb{R}^n$ [**2**, § 10]. $\mu|_F$ denotes the restriction of the measure μ to a set $F \subset \mathbb{R}^n$, defined by $\mu|_F(A) = \mu(F \cap A)$ for $A \subset \mathbb{R}^n$. Notice that, trivially, $\dim_{\mathrm{H}}(\mu) \leqslant \dim_{\mathrm{H}}(\mu|_F)$ whenever F is a Borel set with $\mu(F) > 0$. We will use the notation $\mathcal{H}^s$ to denote the *s-dimensional Hausdorff measure* on $\mathbb{R}^n$. More generally, we denote by $\mathcal{H}_h$ a *generalized Hausdorff measure* constructed using a gauge function $h : (0, r_0) \to (0, \infty)$ [**10**, § 4.9].

Next we will recall the definition of the average homogeneity from [**3**]. If $k \in \mathbb{N}$, then a set $Q \subset \mathbb{R}^n$ is called a *k-adic cube* provided that $Q = [0, k^{-l})^n + k^{-l}z$ for some $l \in \mathbb{N}$ and $z \in \mathbb{Z}^n$. The collection of all k-adic cubes $Q \subset [0,1)^n$ with side length k^{-l} is denoted by $\mathcal{Q}_k^l$. If $Q \in \mathcal{Q}_k^l$ and $t > 0$, then by tQ we denote the cube centred at the same point as Q but with side length tk^{-l}.

Let $k \in \mathbb{N}$ and $I_k = \{1, \dots, k^n\}$. If $\mathtt{i} = (i_1, \dots, i_l) \in I_k^l$ and $i \in I_k$, then we set $\mathtt{i}, i = (i_1, \dots, i_l, i) \in I_k^{l+1}$. Furthermore, if $\mathtt{i} = (i_1, i_2, \dots) \in I_k^\infty := I_k^{\mathbb{N}}$ (or $\mathtt{i} \in I_k^l$) and $j \in \mathbb{N}$ (or $j \leqslant l$), then $\mathtt{i}|_j := (i_1, \dots, i_j) \in I_k^j$. For a given measure μ, we will enumerate k-adic cubes $Q_i \in \mathcal{Q}_k^1$ so that $\mu(Q_i) \leqslant \mu(Q_{i+1})$ whenever $i \in I_k \setminus \{k^n\}$. Given $l \in \mathbb{N}$ and $\mathtt{i} \in I_k^l$, we continue inductively by enumerating the cubes $Q_{\mathtt{i},i} \in \mathcal{Q}_k^{l+1}$ with $Q_{\mathtt{i},i} \subset Q_{\mathtt{i}} \in \mathcal{Q}_k^l$ so that $\mu(Q_{\mathtt{i},i}) \leqslant \mu(Q_{\mathtt{i},i+1})$ whenever $i \in I_k \setminus \{k^n\}$. Bear in mind that this enumeration depends, of course, on the measure. The *(upper) k-average homogeneity of* μ *of order* $i \in I_k$ is defined to be

$$\mathrm{hom}_k^i(\mu) = \limsup_{l \to \infty} \frac{k^n}{l} \sum_{j=1}^{l} \sum_{\mathtt{i} \in I_k^j} \mu(Q_{\mathtt{i},i}).$$

For us it is essential that the Hausdorff dimension of a measure may be bounded above in terms of homogeneity. The following result was obtained in [**3**].

Theorem 2.1. *If μ is a probability measure on $[0,1)^n$ and $\mathrm{hom}_k^i(\mu) \leqslant k^n\eta$ for some $0 \leqslant \eta \leqslant k^{-n}$, then*

$$\dim_{\mathrm{H}}(\mu) \leqslant -\frac{1}{\log k}\biggl(i\eta\log\eta + (1 - i\eta)\log\biggl(\frac{1 - i\eta}{k^n - i}\biggr)\biggr).$$

It is well known that although most measures on $\mathbb{R}^n$ are non-doubling, still 'around typical points most scales are doubling'. This somewhat inexact statement is made quantitative in the following lemma. We follow the convention according to which $c = c(\cdot, \cdot)$ denotes a constant that depends only on the parameters listed inside the parentheses.

Lemma 2.2. *If $n, k \in \mathbb{N}$ and $0 < p < 1$, then there exists a constant $c = c(n,k,p) > 0$ such that for every measure μ on $\mathbb{R}^n$ and for each $\gamma > 0$ we have*

$$\liminf_{l\to\infty} \frac{1}{l} \#\{j \in \{1,\ldots,l\} : \mu(B(x,\gamma k^{-j})) \geqslant c\mu(B(x,\gamma k^{-j+1}))\} \geqslant p$$

for μ-almost every $x \in \mathbb{R}^n$.

Proof. Let $c = k^{-2n/(1-p)}$, fix a measure μ on $\mathbb{R}^n$ and $\gamma > 0$, and denote $N(x,l) = \#\{j \in \{1,\ldots,l\} : \mu(B(x,\gamma k^{-j})) \geqslant c\mu(B(x,\gamma k^{-j+1}))\}$ for $x \in \mathbb{R}^n$ and $l \in \mathbb{N}$. Suppose that $x \in \mathbb{R}^n$ is a point at which

$$\liminf_{l\to\infty} \frac{1}{l} N(x,l) < p.$$

Then there are arbitrarily large integers l such that $N(x,l) < lp$. Hence,

$$\mu(B(x,\gamma k^{-l})) < c^{(1-p)l}\mu(B(x,\gamma))$$

for any such l and, consequently,

$$\begin{aligned}\limsup_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} &\geqslant \limsup_{l\to\infty} \frac{\log c^{(1-p)l} + \log \mu(B(x,\gamma))}{\log \gamma k^{-l}} \\ &= \limsup_{l\to\infty} \frac{2n \log k^{-l}}{\log \gamma k^{-l}} = 2n > n.\end{aligned}$$

But this is possible only in a set of μ-measure zero (see, for example, [**2**, Proposition 10.2]). The claim thus follows. □

3. A general conical density estimate

Our first result is a conical density theorem valid for all measures on $\mathbb{R}^n$. This result is motivated by [**4**, Question 4.3] asking if

$$\limsup_{r\downarrow 0} \inf_{\theta\in S^{n-1}} \frac{\mu(B(x,r) \setminus H(x,\theta,\alpha))}{h(2r)} \geqslant c(n,\alpha) \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{h(2r)}$$

holds μ-almost everywhere for all measures μ on $\mathbb{R}^n$ and all doubling gauge functions h. We shall formulate our result for densities having $\mu(B(x,r))$ in the denominator rather than $h(2r)$ because we believe that these densities are more natural in this general setting. The original question may also be answered in the positive by a slight modification of the proof below.

Theorem 3.1. *If $n \in \mathbb{N}$ and $0 < \alpha \leqslant 1$, then there exists a constant $c = c(n,\alpha) > 0$ so that for every measure μ on $\mathbb{R}^n$ we have*

$$\limsup_{r\downarrow 0} \inf_{\theta\in S^{n-1}} \frac{\mu(B(x,r) \setminus H(x,\theta,\alpha))}{\mu(B(x,r))} > c$$

for μ-almost every $x \in \mathbb{R}^n$.

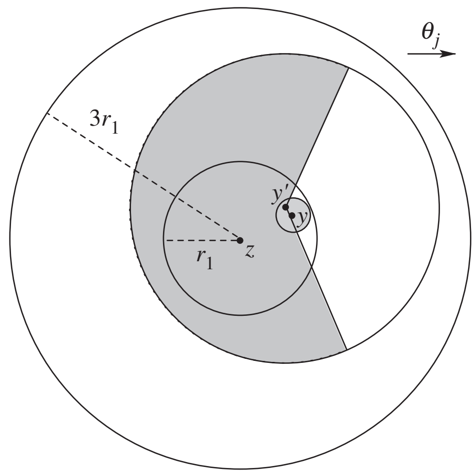

Figure 1. The covering of the set $A \cap B(z, r_1)$ in the proof of Theorem 3.1. The smallest ball is the set D_1 and the shaded sector is the set D_2. The rest of the set is called D_3.

Proof. It is sufficient to consider non-atomic measures since

$$\limsup_{r\downarrow 0} \inf_{\theta\in S^{n-1}} \frac{\mu(B(x,r)\setminus H(x,\theta,\alpha))}{\mu(B(x,r))} = 1$$

if $\mu(\{x\}) > 0$.

Because we want to use only a finite set of directions, we cover the set S^{n-1} with cones $\{H(0,\theta_i,\beta)\}_{i=1}^{K}$, where $\beta = \cos(\arccos(\alpha/2) - \arccos(\alpha))$ and $K = K(n,\alpha) \in \mathbb{N}$. For all $\theta \in S^{n-1}$ there is $i \in \{1,\dots,K\}$ so that $H(x,\theta,\alpha) \subset H(x,\theta_i,\alpha/2)$ for all $x \in \mathbb{R}^n$. Given this, it is enough to show that for all measures μ on $\mathbb{R}^n$ we have

$$\limsup_{r\downarrow 0} \min_{i\in\{1,\dots,K\}} \frac{\mu(B(x,r)\setminus H(x,\theta_i,\alpha/2))}{\mu(B(x,r))} > c = c(\alpha,n) > 0 \tag{3.1}$$

for μ-almost all $x \in \mathbb{R}^n$.

To prove (3.1) we first apply Lemma 2.2 to find a constant $c' < \infty$ depending only on n (choosing $c' = 3^{2n}$ will suffice) so that for all measures μ and for every radius $R > 0$ we have the following: for μ-almost every $x \in \mathbb{R}^n$ there is a scale $r < R$ so that

$$\mu(B(x,3r)) \leqslant c'\mu(B(x,r)).$$

We will prove that (3.1) holds with $c = c(n,\alpha) = (9c'K)^{-1}$. Assume to the contrary that this is not the case. Then we find a non-atomic measure μ and $r_0 > 0$ so that the set

$$A := \left\{x \in \mathbb{R}^n : \min_{i\in\{1,\dots,K\}} \frac{\mu(B(x,r)\setminus H(x,\theta_i,\alpha/2))}{\mu(B(x,r))} < 2c \text{ for every } 0 < r \leqslant r_0\right\}$$

has positive μ-measure. Now A is seen to be a Borel set by standard methods and thus μ-almost all $z \in A$ are μ-density points of A [**10**, Corollary 2.14]. Thus, we may find a point $z \in A$ and a radius $0 < r_1 \leqslant r_0/2$ so that

$$\mu(A \cap B(z,r_1)) \geqslant \tfrac{1}{2}\mu(B(z,r_1)) \tag{3.2}$$

and

$$\mu(B(z,3r_1)) \leqslant c'\mu(B(z,r_1)). \tag{3.3}$$

Now $A \subset \bigcup_{i=1}^{K} A_i$, where

$$A_i := \{x \in A : \mu(B(x,2r_1) \setminus H(x,\theta_i,\alpha/2)) < 2c\mu(B(x,2r_1))\},$$

and thus we may find $j \in \{1,\dots,K\}$ so that

$$\mu(A_j \cap B(z,r_1)) \geqslant K^{-1}\mu(A \cap B(z,r_1)). \tag{3.4}$$

Next take a point y from the closure of $A_j \cap B(z,r_1)$ so that it maximizes the inner product $x \cdot \theta_j$ in the closure of $A_j \cap B(z,r_1)$. Since the measure μ is non-atomic, there is a small radius $r_2 < r_1$ so that

$$\mu(B(y,r_2)) < c'c\mu(B(z,r_1)). \tag{3.5}$$

Now choose any point $y' \in A_j \cap B(z,r_1) \cap B(y,\alpha r_2/3)$ and cover the set $A \cap B(z,r_1)$ with sets D_1, D_2 and D_3 defined by $D_1 = B(y,r_2)$, $D_2 = B(y',2r_1) \setminus H(y',\theta_j,\alpha/2)$ and $D_3 = (A \cap B(z,r_1)) \setminus (D_1 \cup D_2)$ (see Figure 1).

Observe that $D_3 \cap A_j = \emptyset$ and so (3.4) implies

$$\mu(D_3) \leqslant (1-K^{-1})\mu(A \cap B(z,r_1)).$$

Moreover, the inequality (3.5) reads

$$\mu(D_1) < c'c\mu(B(z,r_1)),$$

and with (3.3) and the fact that $y' \in A_j$ we are able to conclude that

$$\mu(D_2) < 2c\mu(B(y',2r_1)) \leqslant 2c\mu(B(z,3r_1)) \leqslant 2c'c\mu(B(z,r_1)).$$

Putting these three estimates together yields

$$\mu(A \cap B(z,r_1)) \leqslant 3c'c\mu(B(z,r_1)) + (1-K^{-1})\mu(A \cap B(z,r_1)),$$

from which we get

$$\mu(A \cap B(z,r_1)) \leqslant 3Kc'c\mu(B(z,r_1)) = \tfrac{1}{3}\mu(B(z,r_1)).$$

This contradicts (3.2) and finishes the proof. □

4. Measures with positive Hausdorff dimension

Suppose that $\mathcal{H}_h$ is a Hausdorff measure constructed using a non-decreasing gauge function $h : (0,r_0) \to (0,\infty)$ and μ is its restriction to some Borel set with finite $\mathcal{H}_h$ measure. There are many works (see, for example, [**5**, **9**, **12**]) that give information on the amount of μ on small cones around $(n-m)$-planes $V \in G(n,n-m)$ when h satisfies suitable assumptions. These results apply when $\mathcal{H}^m$ is purely singular with respect to $\mathcal{H}_h$. In [**4**], similar results are obtained also for many packing-type measures. In this section, we consider general measures with $\dim_{\mathrm{H}}(\mu) > m$ in the same spirit by proving the following result.

Theorem 4.1. *If $n \in \mathbb{N}$, $m \in \{0, \dots, n-1\}$, $s > m$, and $0 < \alpha \leqslant 1$, then there exists a constant $c = c(n, m, s, \alpha) > 0$ so that for every measure μ on $\mathbb{R}^n$ with $\dim_{\mathrm{H}}(\mu) \geqslant s$ we have*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1}, \\ V \in G(n, n-m)}} \frac{\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \alpha))}{\mu(B(x, r))} > c \tag{4.1}$$

for μ-almost every $x \in \mathbb{R}^n$.

We first introduce a couple of geometric lemmas. The first one is proved in [**1**] with the correct asymptotics for $q(n, \alpha)$ as $\alpha \to 0$ (see also [**5**, Lemma 2.1]).

Lemma 4.2. *For each $0 < \alpha \leqslant 1$ there exists $q = q(n, \alpha) \in \mathbb{N}$ such that in any set of q points in $\mathbb{R}^n$ there are always three points x_0, x_1 and x_2 for which $x_1 \in X^+(x_0, \theta, \alpha)$ and $x_2 \in X^+(x_0, -\theta, \alpha)$ for some $\theta \in S^{n-1}$.*

We would like to apply the previous lemma for balls instead of just single points. For this, we will need the following simple lemma.

Lemma 4.3. *For each $0 < \alpha \leqslant 1$ there exists $t = t(\alpha) \geqslant 1$ such that if $x_0, y_0 \in \mathbb{R}^n$ and $r_x, r_y > 0$ are such that $B(x_0, tr_x) \cap B(y_0, tr_y) = \emptyset$ and $y_0 \in X^+(x_0, \theta, \alpha/t)$ for some $\theta \in S^{n-1}$, then*

$$B(y_0, r_y) \subset X^+(x, \theta, \alpha)$$

for all $x \in B(x_0, r_x)$.

Proof. Fix $y \in B(y_0, r_y)$ and $x \in B(x_0, r_x)$. Our aim is to find $t \geqslant 1$ depending only on α, so that under the assumptions of the lemma we have

$$(y - x) \cdot \theta > (1 - \alpha^2)^{1/2} |y - x|.$$

Let $\varepsilon = (1 - (1 - \alpha^2)^{1/2})/2$ and choose $t \geqslant 1$ so large that $(1 - (\alpha/t)^2)^{1/2} \geqslant 1 - \varepsilon$, $(1 - \varepsilon)t - 1 > 0$ and $(1 - \varepsilon)/(1 + 1/t) - 1/(t + 1) > (1 - \alpha^2)^{1/2}$. According to our assumptions, we have

$$|y_0 - x_0| \geqslant t(r_y + r_x), \tag{4.2}$$

$$(y_0 - x_0) \cdot \theta \geqslant (1 - \varepsilon)|y_0 - x_0|. \tag{4.3}$$

Also, we clearly have

$$(y - x) \cdot \theta \geqslant (y_0 - x_0) \cdot \theta - (r_y + r_x) > 0$$

and

$$|y - x| \leqslant |y_0 - x_0| + r_y + r_x.$$

Hence,

$$\frac{(y - x) \cdot \theta}{|y - x|} \geqslant \frac{(y_0 - x_0) \cdot \theta}{|y_0 - x_0| + r_y + r_x} - \frac{r_y + r_x}{|y_0 - x_0| + r_y + r_x}. \tag{4.4}$$

Now (4.2) yields

$$\frac{r_y + r_x}{|y_0 - x_0| + r_y + r_x} \leqslant \frac{1}{t+1}$$

and by using (4.3) and (4.2), we get

$$\frac{|y_0 - x_0| + r_y + r_x}{(y_0 - x_0) \cdot \theta} \leqslant \frac{1}{1-\varepsilon} + \frac{1}{(1-\varepsilon)t}.$$

The proof is finished by combining these estimates with (4.4) and the choice of t. □

The following somewhat technical proposition reduces the proof of Theorem 4.1 to finding a suitable amount of roughly uniformly distributed balls inside $B(x,r)$ all having quite large measure. If this can be done at arbitrarily small scales around typical points, then Theorem 4.1 follows. Below, we shall denote by $\#\mathcal{B}$ the cardinality of a collection $\mathcal{B}$.

Remark 4.4. Observe that $G = G(n, n-m)$ endowed with the metric $d(V,W) = \sup_{x \in V \cap S^{n-1}} \operatorname{dist}(x,W)$ is a compact metric space and

$$\bigcup_{d(W,V)<\alpha} \{x : x \in W\} = X(0,V,\alpha)$$

for all $V \in G$ and $0 < \alpha < 1$ [**12**, Lemma 2.2]. Using the compactness, we may thus choose $K = K(n,m,\alpha) \in \mathbb{N}$ and $(n-m)$-planes $V_1, \ldots, V_K \in G$ so that for each $V \in G$ there exists $j \in \{1, \ldots, K\}$ with

$$X(x,V,\alpha) \supset X(x,V_j,\alpha/2) \tag{4.5}$$

for all $x \in \mathbb{R}^n$.

Proposition 4.5. *Let $m \in \{0, \ldots, n-1\}$, $0 < \alpha \leqslant 1$, $t = t(\alpha/2)$ be the constant of Lemma 4.3 and take $q = q(n-m, \alpha/(2t))$ from Lemma 4.2. Moreover, let $K = K(n,m,\alpha)$ be as in Remark 4.4 and $c > 0$. Suppose that μ is a measure on $\mathbb{R}^n$ and that for μ-almost all $x \in \mathbb{R}^n$ we may find arbitrarily small radii $r > 0$ and a collection $\mathcal{B}$ of sub-balls of $B(x,r)$ with the following properties:*

(i) *the collection $\{2tB : B \in \mathcal{B}\}$ is pairwise disjoint;*

(ii) *$\mu(B) > c\mu(B(x,3r))$ for all $B \in \mathcal{B}$;*

(iii) *if $\mathcal{B}' \subset \mathcal{B}$ with $\#\mathcal{B}' \geqslant \#\mathcal{B}/K$ and $V \in G(n,n-m)$, then there is a translate of V intersecting at least q balls from the collection $\mathcal{B}'$.*

Then

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1}, \\ V \in G(n,n-m)}} \frac{\mu(X(x,r,V,\alpha) \setminus H(x,\theta,\alpha))}{\mu(B(x,r))} > c \tag{4.6}$$

for μ-almost every $x \in \mathbb{R}^n$.

Proof. Let μ be a measure satisfying the assumptions of the proposition and suppose that $(n-m)$-planes $V_1, \ldots, V_K$ are as in Remark 4.4. Our aim is to show that, for μ-almost every $x \in \mathbb{R}^n$, there are arbitrarily small radii $r > 0$ so that for every $j \in \{1, \ldots, K\}$ there is $\zeta = \zeta(x) \in S^{n-1} \cap V_j$ for which

$$\min\{\mu(X^+(x,r,\zeta,\alpha/2)), \mu(X^+(x,r,-\zeta,\alpha/2))\} > c\mu(B(x,r)). \tag{4.7}$$

From this the claim follows easily. Indeed, take $V \in G(n, n-m)$ and choose $V_j \in \{V_1, \ldots, V_K\}$ so that (4.5) holds. Let $\zeta \in V_j \cap S^{n-1}$ satisfy (4.7). Then

$$X^+(x,r,\pm\zeta,\alpha/2) \subset X(x,r,V_j,\alpha/2) \subset X(x,r,V,\alpha)$$

and the claim follows by combining (4.7) with the observation that for all $\zeta', \theta \in S^{n-1}$ we have

$$X^+(x,r,\zeta',\alpha) \cap H(x,\theta,\alpha) = \emptyset \quad \text{or} \quad X^+(x,r,-\zeta',\alpha) \cap H(x,\theta,\alpha) = \emptyset.$$

To prove (4.7), we assume on the contrary that there is a Borel set $F \subset \mathbb{R}^n$ with $\mu(F) > 0$ such that the assumptions (i)–(iii) of Proposition 4.5 hold for every $x \in F$ in some arbitrarily small scales and that, for some $r_0 > 0$ and for every $0 < r < r_0$, there exists $j \in \{1, \ldots, K\}$ so that

$$\mu(X^+(x,r,\zeta,\alpha/2)) \leqslant c\mu(B(x,r)) \quad \text{or} \quad \mu(X^+(x,r,-\zeta,\alpha/2)) \leqslant c\mu(B(x,r)) \tag{4.8}$$

for all $\zeta \in S^{n-1} \cap V_j$. Now choose a μ-density point x_1 of F and a radius $0 < r_1 < r_0/3$ so that

$$\mu(B(x_1,r) \setminus F) < c\mu(B(x_1,r)) \leqslant c\mu(B(x_1,3r)) \tag{4.9}$$

for all $0 < r < r_1$. Next we choose a radius $0 < r < r_1$ and a collection of balls $\mathcal{B}$ inside $B(x_1,r)$ satisfying the assumptions (i)–(iii) of Proposition 4.5. Then we let

$$F_j = \{x \in B(x_1,r) \cap F : \text{(4.8) holds with this } r \text{ for all } \zeta \in S^{n-1} \cap V_j\}.$$

for $j \in \{1, \ldots, K\}$. According to (4.9) each ball of $\mathcal{B}$ contains points of F and hence there is at least one $j \in \{1, \ldots, K\}$ so that not less than $\#\mathcal{B}/K$ balls among $\mathcal{B}$ contain points of F_j. Fix such a j, and let $\mathcal{B}' = \{B \in \mathcal{B} : F_j \cap B \neq \emptyset\}$. Then Proposition 4.5 (iii) implies that we may find $z \in \mathbb{R}^n$ and q different balls $B_1, \ldots, B_q \in \mathcal{B}'$ so that they all intersect the affine $(n-m)$-plane $V_j + z$. According to Proposition 4.5 (i) and Lemmas 4.2 and 4.3, we may choose three balls B^0, B^1, B^2 among the balls $B_1, \ldots, B_q$ and a point $x_0 \in F_j \cap B^0$ so that for some $\theta \in S^{n-1} \cap V_j$ we have

$$B^1 \subset X^+(x_0,\theta,\alpha/2) \quad \text{and} \quad B^2 \subset X^+(x_0,-\theta,\alpha/2).$$

But this contradicts (4.8) since $\min\{\mu(B^1), \mu(B^2)\} > c\mu(B(x_1,3r)) \geqslant c\mu(B(x_0,2r))$ by Proposition 4.5 (ii). □

To complete the proof of Theorem 4.1, we need to find collections $\mathcal{B}$ of balls as in the previous proposition. To that end, we first work with cubes (instead of balls) and use Theorem 2.1.

Lemma 4.6. *For any $n \in \mathbb{N}$, $m \in \{0,\dots,n-1\}$, $s > m$, $M \in \mathbb{N}$, $\tau \geqslant 1$ and $k > M^{1/(s-m)}$ there exist constants $c = c(n,m,s,M,\tau,k) > 0$ and $0 < p = p(n,m,s,M,\tau,k) < 1$ satisfying the following. For every measure μ on $[0,1)^n$ with $\dim_{\mathrm{H}}(\mu) \geqslant s$ and for μ-almost every $x \in [0,1)^n$,*

$$\limsup_{l\to\infty} \frac{1}{l}\#\{j \in \{1,\dots,l\} : \mu(Q_{\mathtt{i},k^n-Mk^m}) > c\mu(\tau Q_{\mathtt{i}}),\quad \text{where } \mathtt{i} \in I_k^j \text{ is such that } x \in Q_{\mathtt{i}}\} > p. \tag{4.10}$$

Here we use the enumeration of the k-adic cubes introduced in § 2.

Proof. Since $\log(Mk^m)/\log(k) < s$, it follows by an easy calculation that we may choose a number $c = c(n,m,s,M,\tau,k) > 0$ such that $0 < \eta := 3c(3\sqrt{n}\tau+2)^n < k^{-n}$ and

$$-\frac{1}{\log k}\left((k^n-Mk^m)\eta\log\eta + \left(1-(k^n-Mk^m)\eta\right)\log\left(\frac{1-(k^n-Mk^m)\eta}{Mk^m}\right)\right) < s. \tag{4.11}$$

We will prove the claim with this choice of c, and with $p = c(3\sqrt{n}\tau+2)^n$. Suppose to the contrary that there is a Borel set $F \subset [0,1)^n$ with $\mu(F) > 0$ such that (4.10) does not hold for any point of F. Consider the restriction measure $\mu|_F$. In order to use Theorem 2.1, we scale our original measure so that $\mu(F) = 1$. Note that this scaling does not affect the dimension of μ or the condition (4.10). It is sufficient to show that

$$\hom_k^{k^n-Mk^m}(\mu|_F) \leqslant 3ck^n(3\sqrt{n}\tau+2)^n \tag{4.12}$$

since this would imply $\dim_{\mathrm{H}}(\mu) \leqslant \dim_{\mathrm{H}}(\mu|_F) < s$ by Theorem 2.1 and (4.11). In order to calculate $\hom_k^{k^n-Mk^m}(\mu|_F)$, we need to enumerate the k-adic cubes in terms of $\mu|_F$ rather than in terms of μ. We denote cubes enumerated in terms of $\mu|_F$ by $Q'_{\mathtt{i}}$.

Observe that if $Q \in \mathcal{Q}_k^j$, then any ball centred at Q with radius $\sqrt{n}\tau k^{-j}$ contains the cube τQ and is contained in the cube $3\sqrt{n}\tau Q$. If $x \in F$ is a μ-density point of F, then $\mu(B(x,r)) \leqslant 2\mu(F\cap B(x,r))$ for all sufficiently small $r > 0$. If $j \in \mathbb{N}$ is large enough and $\mu(Q_{\mathtt{i},k^n-Mk^m}) \leqslant c\mu(\tau Q_{\mathtt{i}})$, where $\mathtt{i} \in I_k^j$ is such that $x \in Q_{\mathtt{i}}$, then also

$$\begin{aligned}
\mu(F\cap Q_{\mathtt{i},i}) &\leqslant \mu(Q_{\mathtt{i},k^n-Mk^m}) \\
&\leqslant c\mu(\tau Q_{\mathtt{i}}) \leqslant c\mu(B(x,\sqrt{n}\tau k^{-j})) \\
&\leqslant 2c\mu(F\cap B(x,\sqrt{n}\tau k^{-j})) \\
&\leqslant 2c\mu(F\cap 3\sqrt{n}\tau Q_{\mathtt{i}})
\end{aligned}$$

for all $i \in \{1,\dots,k^n-Mk^m\}$ and so also

$$\mu(F\cap Q'_{\mathtt{j},k^n-Mk^m}) \leqslant 2c\mu(F\cap 3\sqrt{n}\tau Q'_{\mathtt{j}}), \tag{4.13}$$

where $Q'_{\mathtt{j}} = Q_{\mathtt{i}}$.

We define

$$E_k^j = \{\mathtt{i} \in I_k^j : \mu(F\cap Q'_{\mathtt{i},k^n-Mk^m}) \leqslant 2c\mu(F\cap 3\sqrt{n}\tau Q'_{\mathtt{i}})\}$$

for $j \in \mathbb{N}$ and

$$N(x,l) = \#\{j \in \{1,\dots,l\} : \mathtt{i}|_j \in E_k^j \text{ where } \mathtt{i} \in I_k^l \text{ is such that } x \in Q'_{\mathtt{i}}\}$$

for $x \in [0,1)^n$ and $l \in \mathbb{N}$. It follows from the choice of the set F and (4.13) that

$$\liminf_{l\to\infty} \frac{1}{l} N(x,l) \geqslant 1 - p$$

for μ-almost every $x \in F$. Since $N(x,l)$ is constant on $Q'_{\mathtt{i}}$ whenever $\mathtt{i} \in I_k^l$, this implies

$$\liminf_{l\to\infty} \frac{1}{l} \sum_{j=1}^{l} \sum_{\mathtt{i}\in E_k^j} \mu(F \cap Q'_{\mathtt{i}}) = \liminf_{l\to\infty} \frac{1}{l} \int_F N(x,l) d\mu(x) \geqslant 1 - p$$

by Fatou's lemma and, consequently,

$$\limsup_{l\to\infty} \frac{1}{l} \sum_{j=1}^{l} \sum_{\mathtt{i}\notin E_k^j} \mu(F \cap Q'_{\mathtt{i}}) \leqslant p.$$

Moreover,

$$\sum_{\mathtt{i}\in I_k^j} \mu(F \cap 3\sqrt{n}\tau Q'_{\mathtt{i}}) \leqslant (3\sqrt{n}\tau + 2)^n$$

for every $j \in \mathbb{N}$, because each cube $Q \in \mathcal{Q}_k^j$ intersects at most $(3\sqrt{n}\tau + 2)^n$ larger cubes $3\sqrt{n}\tau\tilde{Q}$, where $\tilde{Q} \in \mathcal{Q}_k^j$. Combining the previous two estimates and the choice of p, we now obtain

$$\begin{aligned}
&\hom_k^{k^n - Mk^m}(\mu|_F) \\
&\quad = \limsup_{l\to\infty} \frac{k^n}{l} \sum_{j=1}^{l} \Bigg(\sum_{\mathtt{i}\in E_k^j} \mu(F \cap Q'_{\mathtt{i},k^n-Mk^m}) + \sum_{\mathtt{i}\notin E_k^j} \mu(F \cap Q'_{\mathtt{i},k^n-Mk^m}) \Bigg) \\
&\quad \leqslant \limsup_{l\to\infty} \frac{k^n}{l} \sum_{j=1}^{l} \sum_{\mathtt{i}\in I_k^j} 2c\mu(F \cap 3\sqrt{n}\tau Q'_{\mathtt{i}}) + \limsup_{l\to\infty} \frac{k^n}{l} \sum_{j=1}^{l} \sum_{\mathtt{i}\notin E_k^j} \mu(F \cap Q'_{\mathtt{i},k^n-Mk^m}) \\
&\quad \leqslant 2ck^n(3\sqrt{n}\tau + 2)^n + pk^n \\
&\quad = 3ck^n(3\sqrt{n}\tau + 2)^n.
\end{aligned}$$

This completes the proof. □

To finish the proof of Theorem 4.1, we just need to combine the previous lemma and Proposition 4.5 and show how cubes may be replaced by balls. We will choose the number of cubes $Q_{\mathtt{i},i}$ with $\mu(Q_{\mathtt{i},i}) > c\mu(\tau Q_{\mathtt{i}})$ (using the notation of Lemma 4.6) large enough so that we are able to choose sufficiently many appropriately separated balls $Q_{\mathtt{i},i} \subset B_i \subset \tau Q_{\mathtt{i}}$. In order to find a ball containing $\tau Q_{\mathtt{i}}$ with comparable measure, we need to work on a doubling scale for the measure μ. For this, we will use Lemma 2.2.

Proof of Theorem 4.1. Observe that without loss of generality we may assume μ to be a probability measure with $\text{spt}(\mu) \subset [0,1)^n$. Let $t = t(\alpha/2) \geqslant 1$ be the constant of Lemma 4.3 and set $q = q(n-m, \alpha/(2t))$ from Lemma 4.2. Moreover, let $K = K(n,m,\alpha)$ be as in Remark 4.4 and choose $M = M(n,m,\alpha) \in \mathbb{N}$ so that $M \geqslant \text{vol}(n)(4t+2)^n n^{n/2} 8^m Kq$, where $\text{vol}(n)$ is the n-dimensional volume of the unit ball.

If $Q \in \mathcal{Q}_k^j$ for some $j, k \in \mathbb{N}$ and $\tau = 6\sqrt{n}$, it follows that

$$2Q \subset B(x, 2\sqrt{n}k^{-j}) \subset \tau Q, \tag{4.14}$$

$$B(y, \sqrt{n}k^{-j-1}) \subset B(x, 2\sqrt{n}k^{-j}) \tag{4.15}$$

for every $x, y \in Q$. Choose $k = k(n,m,s,\alpha) \in \mathbb{N}$ so that $k > \max\{M^{1/(s-m)}, 3\}$ and let $c_1 = c(n,m,s,M,\tau,k) > 0$ and $0 < p = p(n,m,s,M,\tau,k) < 1$ be as in Lemma 4.6 and let $c_2 = c(n,k,1-p/2) > 0$ be the constant of Lemma 2.2. Combining these lemmas, it follows that for μ-almost all $x \in [0,1)^n$ there are arbitrarily large $j \in \mathbb{N}$ and $\mathtt{i} \in I_k^j$ with $x \in Q_{\mathtt{i}}$ such that with $r = 2\sqrt{n}k^{-j}$ we have

$$\mu(B(x,r)) \geqslant c_2 \mu(B(x, 2\sqrt{n}k^{-j+1})), \tag{4.16}$$

$$\mu(Q_{\mathtt{i},k^n - Mk^m}) > c_1 \mu(\tau Q_{\mathtt{i}}). \tag{4.17}$$

To obtain (4.16), we use Lemma 2.2 with $\gamma = 2\sqrt{n}$. To complete the proof, the only thing to check is that with any such x and r we may find a collection $\mathcal{B}$ satisfying the assumptions (i)–(iii) of Proposition 4.5.

Combining (4.17), (4.14) and (4.16) and recalling that $k \geqslant 3$, we have

$$\mu(Q_{\mathtt{i},i}) > c_1 \mu(B(x,r)) \geqslant c_1 c_2 \mu(B(x,3r)) \tag{4.18}$$

for every $i \in \{k^n - Mk^m, \ldots, k^n\}$. Let $B_i = B(y_i, \sqrt{n}k^{-j-1})$, where y_i is the centre point of $Q_{\mathtt{i},i}$. Then $\mu(B_i) > c_1 c_2 \mu(B(x,3r))$ and $B_i \subset B(x,r)$ by (4.15). By a simple volume argument, we have

$$\#\{j : 2tB_i \cap 2tB_j \neq \emptyset\} \leqslant \text{vol}(n)(4t+2)^n n^{n/2}$$

for every i. Consequently, there is a sub-collection $\mathcal{B}$ of the collection $\{B_i\}$ containing at least $8^m Kqk^m$ balls so that the collection $\{2tB : B \in \mathcal{B}\}$ is pairwise disjoint and $\mu(B) > c_1 c_2 \mu(B(x,3r))$ for all $B \in \mathcal{B}$. To check that Proposition 4.5 (iii) also holds, choose any sub-collection $\mathcal{B}'$ of $\mathcal{B}$ with $\#\mathcal{B}' \geqslant \#\mathcal{B}/K \geqslant 8^m qk^m$ and fix $V \in G(n, n-m)$. Since the m-dimensional ball $\text{proj}_{V^\perp}(B(x,r))$ may be covered by $8^m k^m$ balls of radius $\sqrt{n}k^{-j-1}$, it follows that some translate of V must hit at least q balls from the collection $\mathcal{B}'$. Here $V^\perp$ denotes the orthogonal complement of V. Thus, we have verified the assumptions of Proposition 4.5 and the claim follows with $c = c(n,m,s,\alpha) = c_1 c_2$. □

Remark 4.7. (i) Our method to prove Theorem 4.1 could be pushed further to obtain the following quantitative upper conical density theorem: under the assumptions of Theorem 4.1, we have

$$\limsup_{l \to \infty} \frac{1}{l} \#\left\{ j \in \{1, \ldots, l\} : \inf_{\substack{\theta \in S^{n-1}, \\ V \in G(n,n-m)}} \frac{\mu(X(x, 2^{-j}, V, \alpha) \setminus H(x, \theta, \alpha))}{\mu(B(x, 2^{-j}))} > c \right\} > p$$

for μ-almost all points $x \in \mathbb{R}^n$ with some constants $c = c(\alpha, s, n, m) > 0$ and $p = p(\alpha, s, n, m) > 0$.

(ii) One could also apply Mattila's result [**9**, Theorem 3.1] to obtain results analogous to Theorem 4.1. More precisely, the quantity

$$\inf_{\substack{\theta \in S^{n-1}, \\ V \in G(n,n-m)}} \frac{\mu(X(x,r,V,\alpha) \setminus H(x,\theta,\alpha))}{\mu(B(x,r))}$$

can be replaced by

$$\inf_{C} \frac{\mu(C_x \cap B(x,r))}{\mu(B(x,r))},$$

where the infimum is over all Borel sets $C \subset G(n, n-m)$ with $\gamma(C) > \delta > 0$. Here $C_x = \bigcup_{V \in C}(V + x)$ and γ is the natural isometry invariant Borel probability measure on the Grassmannian $G(n, n-m)$. The constant $c > 0$ obtained then depends on n, m, s and δ.

Thus, using Mattila's method would yield more general results in the sense that the cones $X(x, V, \alpha)$ could be replaced by the more general cones C_x. On the other hand, our method also allows consideration of the non-symmetric cones $X(x, V, \alpha) \setminus H(x, \theta, \alpha)$ and may be used to obtain quantitative estimates as in Remark 4.7 (i).

5. Examples and open problems

Inspecting the proof of Proposition 4.5, we see that the assumptions of Theorem 4.1 imply that we may, in fact, find directions $\theta_{x,V} \in S^{n-1} \cap V$, depending on the point x, such that

$$\limsup_{r \downarrow 0} \inf_{V \in G(n,n-m)} \frac{\min\{\mu(X^+(x,r,\theta_{x,V},\alpha)), \mu(X^+(x,r,-\theta_{x,V},\alpha))\}}{\mu(B(x,r))} > c \tag{5.1}$$

for μ-almost all $x \in \mathbb{R}^n$. If $m = 0$, we do not know if the assumption $\dim_{\mathrm{H}}(\mu) > 0$ is necessary or not.

Question 5.1. Given $\alpha > 0$ and $n \in \mathbb{N}$, does there exist a constant $c(n, \alpha) > 0$ such that for all non-atomic measures μ on $\mathbb{R}^n$ one could pick $\theta = \theta(x) \in S^{n-1}$ for μ-almost all $x \in \mathbb{R}^n$ so that

$$\limsup_{r \downarrow 0} \frac{\min\{\mu(X^+(x,r,\theta,\alpha)), \mu(X^+(x,r,-\theta,\alpha))\}}{\mu(B(x,r))} > c?$$

Remark 5.2. (i) A positive answer would also improve Theorem 3.1. However, the question is relevant only for $n \geqslant 2$. If $n = 1$, there is no difference between the above question and Theorem 3.1.

(ii) Examples 5.4 and 5.5 show that we cannot hope to obtain (4.1) if the dimension of μ is m, even if μ is purely unrectifiable (see the definition before Example 5.5). Thus, Question 5.1 is really only about non-atomic measures with zero Hausdorff dimension.

The following example shows why we cannot apply Proposition 4.5 to answer Question 5.1. For simplicity, we will work on $\mathbb{R}$, although similar constructions also work in higher dimensions.

Example 5.3. There is a non-atomic measure μ on $\mathbb{R}$ so that it fails to satisfy the assumptions of Proposition 4.5 with $m = 0$ for all $c > 0$.

Construction. We will construct the measure μ on $[0, 1)$. Our aim is to show that there is no constant $c > 0$, so that for μ-almost all $x \in [0, 1)$ there would be arbitrarily small radii $r > 0$ such that we could find intervals $I_1, \ldots, I_6 \subset (x - r, x + r)$ for which

$$3I_i \cap 3I_j = \emptyset \quad \text{whenever } i \neq j, \tag{5.2}$$

$$\mu(I_i) > c\mu(x - 3r, x + 3r) \quad \text{for all } i. \tag{5.3}$$

To construct μ, we simply take any sequence $0 < q_i < \frac{1}{2}$ so that

$$\sum_{i=1}^{\infty} q_i = \infty$$

and $q_i \downarrow 0$ as $i \to \infty$. Then we construct a binomial-type measure using the weights q_i and $p_i = 1 - q_i$. Let $\mu([0, \frac{1}{2})) = p_1$ and $\mu([\frac{1}{2}, 1)) = q_1$. If $i \in \mathbb{N}$ and $J \in \mathcal{Q}_2^i$, then for $I_1, I_2 \in \mathcal{Q}_2^{i+1}$, where $I_1 \subset J$ is the left-hand side subinterval and $I_2 \subset J$ is the right-hand side subinterval, we set $\mu(I_1) = p_{i+1}\mu(J)$ and $\mu(I_2) = q_{i+1}\mu(J)$. This construction extends to a measure by standard methods.

Suppose there is a constant $c > 0$ for which (5.2) and (5.3) hold. Choose $i_0 \in \mathbb{N}$ so that

$$q_i < c/3 \quad \text{for all } i > i_0. \tag{5.4}$$

We may assume that (5.2) and (5.3) are valid for $I_1, \ldots, I_6 \subset I := (x - r, x + r) \subset [0, 1)$ with $r \ll 2^{-i_0}$. Choose $l \in \mathbb{N}$ for which $2^{-l-1} \leqslant 2r < 2^{-l}$. Then I intersects at most three dyadic intervals of length 2^{-l-1} and one of these dyadic intervals, say J, must contain at least two of the intervals $I_1, \ldots, I_6$, say I_1 and I_2. Now $J \subset 3I$ so $\mu(I_1), \mu(I_2) > c\mu(3I) \geqslant c\mu(J)$.

Let $J_0 \subset J$ be the largest dyadic subinterval of J with the same left-hand side end point as J for which

$$\mu(J_0) < c\mu(J). \tag{5.5}$$

Let y be the right-hand side end point of J_0 and let $J_1, \ldots, J_k$ be the maximal dyadic subintervals of J which do not intersect J_0. So $J = J_0 \cup J_1 \cup \cdots \cup J_k$ and $J_i \cap J_j = \emptyset$ whenever $i \neq j$. It follows from the construction of μ and (5.4) that $\mu(J_i) \leqslant \frac{1}{2}c\mu(J)$ for all $i \geqslant 1$. So if $y \notin I_1$, then $I_1 \cap J_0 = \emptyset$ by (5.5), and I_1 has to intersect at least three of the intervals $J_1, \ldots, J_k$. Then $J_i \subset I_1$ for at least one $i \geqslant 1$. Since $J_0 \subset 3J_i$ for all i, it follows that also $J_0 \subset 3I_1$. In particular, $y \in 3I_1$ in any case. By the same argument, we also have $y \in 3I_2$, so $3I_1 \cap 3I_2 \neq \emptyset$ contrary to (5.2).

Observe that one may replace 3 in (5.3) by any number $a > 1$, but then 6 (the number of the chosen subintervals) needs to be replaced by $n = n(a) \in \mathbb{N}$. □

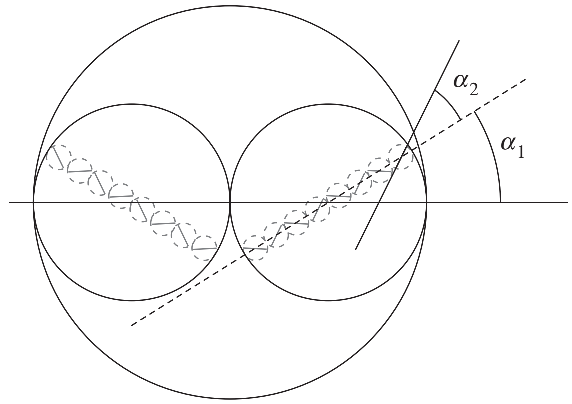

Figure 2. The construction of the set A in Example 5.4.

To finish the paper, we give the examples mentioned in Remark 5.2 (ii). Suppose that $A \subset \mathbb{R}^n$ is purely m-unrectifiable and satisfies $0 < \mathcal{H}^m(A) < \infty$. We refer the reader to [**10**] for the basic properties of unrectifiable sets. If $0 < \alpha < 1$ and $V \in G(n, n-m)$, it is well known that

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^m(A \cap X(x,r,V,\alpha))}{(2r)^m} > c(m,\alpha) > 0 \tag{5.6}$$

for $\mathcal{H}^m$-almost all $x \in A$. The following example, answering [**4**, Question 4.2], shows that this cannot be improved to

$$\limsup_{r \downarrow 0} \inf_{V \in G(n,n-m)} \frac{\mathcal{H}^m(A \cap X(x,r,V,\alpha))}{(2r)^m} > c(m,\alpha) > 0.$$

Example 5.4. There exists a purely 1-unrectifiable compact set $A \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(A) < \infty$ so that for every $0 < \alpha \leqslant 1$,

$$\lim_{r \downarrow 0} \inf_{\ell \in G(2,1)} \frac{\mathcal{H}^1(A \cap X(x,r,\ell,\alpha))}{2r} = 0 \tag{5.7}$$

for every $x \in A$.

Construction. We construct the set A using a nested sequence of compact sets. The first set A_0 is just the unit ball $B(0,1)$. To define the rest of the construction sets, we apply the ideas found, for example, in [**8**, § 5.3] and [**11**, § 5.8].

Define a collection of mappings $f_{i,j}$ with $i \in \mathbb{N}$ and $j \in \{1, \ldots, 2i^2\}$ as

$$f_{i,j}(x,y) = \frac{1}{2i^2}((\cos(\alpha_i)x + 2j - 2i^2 - 1) - (-1)^j \sin(\alpha_i)y, (-1)^j \sin(\alpha_i)x + \cos(\alpha_i)y),$$

where $\alpha_i = 1/\sqrt{i}$. Then define sets A_n for $n \in \mathbb{N}$, as

$$A_n = \bigcup_{\substack{i \in \{1,\ldots,n\}, \\ j_i \in \{1,\ldots,2i^2\}}} f_{1,j_1} \circ \cdots \circ f_{n,j_n}(A_0).$$

Finally, set $A = \bigcap_{n=1}^{\infty} A_n$. See Figure 2 to see the first three steps, A_0, A_1 and A_2, of the construction. We refer to the radius of a step-n construction ball as R_n. That is $R_0 = 1$ and $R_n = R_{n-1}/2n^2$ for $n \geqslant 1$.

Let us verify that the set A admits the desired properties. It is evident from the construction that $A \subset B(0,1)$ is a compact set with $0 < \mathcal{H}^1(A) \leqslant 1$. The upper bound is trivial, as the sum of the diameters of level-n construction balls is always 1. If $F \subset B(0,1)$, then there exist n and a collection $\mathcal{B}$ of level-n construction balls covering $F \cap A$ so that

$$\sum_{B \in \mathcal{B}} \operatorname{diam}(B) < 10 \operatorname{diam}(F).$$

This gives the lower bound. Moreover, we have $\mathcal{H}^1(A \cap B_n) = R_n \mathcal{H}^1(A)$ for each construction ball B_n of level n. For each $x \in A$ there is a unique address $a(x) = (a_1(x), a_2(x), \dots)$ so that $a_i(x) \in \{1, \dots, 2i^2\}$ and

$$\{x\} = \bigcap_{i=1}^{\infty} f_{1,a_1(x)} \circ \cdots \circ f_{i,a_i(x)}(A_0).$$

By Kolmogorov's Zero-One Law and the three-series criteria (see, for example, [**6**]), the series

$$\sum_{i=1}^{n} (-1)^{a_i(x)} \alpha_i$$

diverges for $\mathcal{H}^1$-almost every $x \in A$. Take such a point x and fix an angle $\beta \in [0, 2\pi]$. Since $\alpha_i \downarrow 0$ as $i \to \infty$, there exists $\varepsilon > 0$ so that

$$\limsup_{n\to\infty} \min_{k \in \mathbb{Z}} \left| \beta - \sum_{i=1}^{n} (-1)^{a_i(x)} \alpha_i + k\pi \right| > 4\varepsilon.$$

Let ℓ_β be the line with an angle β. We will show that

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^1(A \cap B(x,r) \setminus X(x, \ell_\beta, \varepsilon))}{r} > 0. \tag{5.8}$$

This means that ℓ_β is not an approximate tangent of A at x and thus A is purely 1-unrectifiable (see, for example, [**10**, Corollary 15.20]). Take $n \in \mathbb{N}$ large enough so that

$$\min_{k \in \mathbb{Z}} \left| \beta - \sum_{i=1}^{n} (-1)^{a_i(x)} \alpha_i + k\pi \right| > 2\varepsilon.$$

Since all the $2n^2$ level-n construction balls inside the ball $f_{1,a_1(x)} \circ \cdots \circ f_{n-1,a_{n-1}(x)}(A_0)$ hit the line from x with direction

$$\sum_{i=1}^{n} (-1)^{a_i(x)} \alpha_i,$$

there exists K depending only on ε (it suffices to take $K > 10/\varepsilon$) so that

$$\#\{m : B_m \cap X(x, R_{n-1}, \ell_\beta, \varepsilon) \neq \emptyset\} \leqslant K,$$

where $B_m = f_{1,a_1(x)} \circ \cdots \circ f_{n-1,a_{n-1}(x)} \circ f_{n,m}(A_0)$. This yields an adequate surplus of balls outside the cone $X(x, \ell_\beta, \varepsilon)$, giving

$$\frac{\mathcal{H}^1(A \cap B(x, R_{n-1}) \setminus X(x, \ell_\beta, \varepsilon))}{R_{n-1}} \geqslant \frac{2n^2 - K}{2n^2} \mathcal{H}^1(A),$$

and therefore (5.8) holds.

It remains to verify that (5.7) holds. Let $x \in A$ and $0 < \alpha \leqslant 1$. First, observe from the construction that with any $n \in \mathbb{N}$ and $y \in A \setminus (f_{1,a_1(x)} \circ \cdots \circ f_{n-1,a_{n-1}(x)}(A_0))$ we have

$$\operatorname{dist}(y, x) \geqslant (1 - \cos(\alpha_n))R_{n-1} \geqslant \frac{R_{n-1}}{4n} = \frac{2n^2 R_n}{4n} = \frac{nR_n}{2}.$$

Let $0 < r < 1$ and choose the $n \in \mathbb{N}$ for which $nR_n \leqslant 2r < (n-1)R_{n-1}$. Let ℓ be the line perpendicular to the direction

$$\sum_{i=1}^{n-1} (-1)^{a_i(x)} \alpha_i.$$

Now there exist numbers $M, n_0 \in N$ depending only on α (letting $M > 10/\alpha$ and n_0 so that $\alpha_{n_0 - 1} < \alpha/10$ will suffice) so that if $n \geqslant n_0$, then

$$\#\{m : B_m \cap X(x, r, \ell, \alpha) \neq \emptyset\} \leqslant M,$$

where the B_m denote the construction balls of level n. Thus,

$$\frac{\mathcal{H}^1(A \cap X(x, r, \ell, \alpha))}{2r} \leqslant \frac{MR_n \mathcal{H}^1(A)}{nR_n} = \frac{M}{n} \mathcal{H}^1(A) \to 0,$$

as $r \downarrow 0$. □

A measure μ on $\mathbb{R}^n$ is called *purely m-unrectifiable* if $\mu(A) = 0$ for all m-rectifiable sets $A \subset \mathbb{R}^n$. The following example shows that a result analogous to (5.6) does not hold for arbitrary purely m-unrectifiable measures on $\mathbb{R}^m$.

Example 5.5. There exist $\ell \in G(2,1)$ and a measure μ on $\mathbb{R}^2$ so that μ is purely 1-unrectifiable and, for every $0 < \alpha < 1$,

$$\lim_{r \downarrow 0} \frac{\mu(X(x, r, \ell, \alpha))}{\mu(B(x, r))} = 0 \tag{5.9}$$

for μ-almost all $x \in \mathbb{R}^2$.

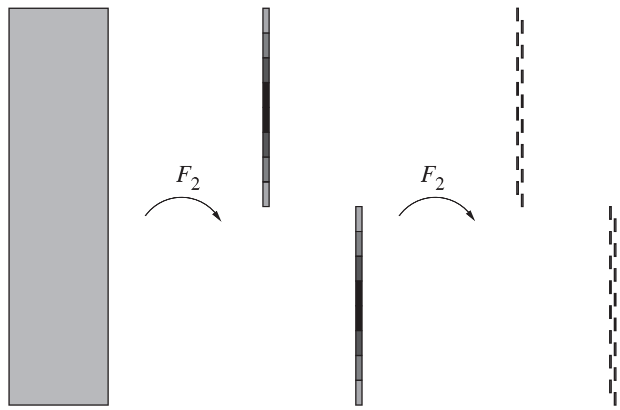

Figure 3. The distribution of the measure with map F_2 in Example 5.5.

Construction. We construct the measure μ using families of maps

$$\{f^i_{k,h} : k \in \{0, \dots, i-1\} \text{ and } h \in \{0, \dots, 2i^2 - 1\}\}_{i=1}^{\infty}$$

with

$$f^i_{k,h}((x,y)) = \left(\frac{(-1)^k i + x}{2i^3}, \frac{2ki^2 + h + y}{2i^3} \right)$$

for every $i \in \{2, 3, \dots\}$, $k \in \{0, \dots, i-1\}$ and $h \in \{0, \dots, 2i^2 - 1\}$.

With $\{f^i_{k,h}\}_{k,h}$, define F_i mapping a measure ν on $\mathbb{R}^2$ to a measure $F_i(\nu)$, so that for every Borel set $A \subset \mathbb{R}^2$ we get

$$F_i(\nu)(A) = \sum_{k=0}^{i-1} \sum_{h=0}^{2i^2-1} C_i (2i)^{-|h-i^2+1/2|} \nu((f^i_{k,h})^{-1}(A)), \tag{5.10}$$

where the constant C_i is chosen so that

$$\sum_{k=0}^{i-1} \sum_{h=0}^{2i^2-1} C_i (2i)^{-|h-i^2+1/2|} = 1.$$

Applying the map F_i divides the measure into i vertical strips. These strips correspond to the index k in the mappings $f^i_{k,h}$. Inside the strips, the measure is divided into $2i^2$ blocks using the index h. The measure is concentrated near the centres of the strips by giving different weights to the maps $f^i_{k,h}$ with different values of h. See Figure 3 to get the idea of the distribution of mass under map F_i.

Let $N_1 = 0$ and for $i \in \{2, 3, \dots\}$ let N_i be the smallest integer so that

$$\left(1 - \frac{C_i}{8(2i)^{i^2-3/2}}\right)^{N_i} < \frac{1}{2}. \tag{5.11}$$

Integers N_i determine how many times we have to use map F_i when constructing the measure μ in order to make the resulting measure unrectifiable. With these numbers define $(I_j)_{j=1}^{\infty}$ with

$$I_{p+\sum_{i=1}^{t-1} N_i} = t$$

for every $t \in \{2, 3, \dots\}$ and $p \in \{1, \dots, N_t\}$. Also let $M_j = \prod_{i=1}^{j}(2I_i^3)$. Finally, define μ to be the weak limit of

$$F_{I_1} \circ F_{I_2} \circ \cdots \circ F_{I_m}(\mu_0)$$

as $m \to \infty$. Here μ_0 is any compactly supported Borel probability measure on $\mathbb{R}^2$. (Take, for example, $\mathcal{H}^1$ restricted to $\{0\} \times [0, 1]$.) With $i \in \mathbb{N}$, $k \in \{1, \dots, M_{i-1}I_i\}$ and $h \in \{1, \dots, 2I_i^2\}$ define strips

$$S_{i,k} = \operatorname{spt}(\mu) \cap \left(\mathbb{R} \times \left[\frac{2(k-1)I_i^2}{M_i}, \frac{2kI_i^2}{M_i}\right]\right)$$

and blocks

$$B_{i,k,h} = \operatorname{spt}(\mu) \cap \left(\mathbb{R} \times \left[\frac{2(k-1)I_i^2 + h - 1}{M_i}, \frac{2kI_i^2 + h}{M_i}\right]\right).$$

To prove the unrectifiability, let us first look at vertical curves. Let γ be a C^1-curve in $\mathbb{R}^2$ so that

$$\left|\frac{\partial \gamma}{\partial y}\right| \geqslant \frac{|\gamma'|}{3}.$$

Take $i \in \mathbb{N}$. Now, for any $k \in \{1, \dots, I_{i+1} - 1\}$ and $t \in \{0, \dots, M_i - 1\}$, either

$$\gamma \cap B_{i+1, 2I_{i+1}^3 t + k, 2I_{i+1}^2} = \emptyset \quad \text{or} \quad \gamma \cap B_{i+1, 2I_{i+1}^3 t + k + 1, 1} = \emptyset.$$

This means that when we look at two consecutive strips $S_{i+1, 2I_{i+1}^3 t + k}$ and $S_{i+1, 2I_{i+1}^3 t + k + 1}$ we see that the curve γ cannot meet both the uppermost block of the lower strip and the lowest block of the upper strip. This is because vertically these blocks are next to each other, but horizontally the distance is roughly at least I_{i+1} times the width of the block. Hence, the curve γ misses more than one-quarter of all the end blocks of the strips of the level I_{i+1} construction step. Therefore, by iterating and using inequality (5.11), we get

$$\begin{aligned}
\mu(\gamma) &\leqslant \prod_{i=1}^{M} \left(1 - \frac{I_{i+1} C_{I_{i+1}} (2I_{i+1})^{-I_{i+1}^2 + 1/2}}{4}\right) \\
&\leqslant \prod_{m=2}^{I_M - 1} \left(1 - \frac{C_m}{8(2m)^{m^2 - 3/2}}\right)^{N_m} \\
&< 2^{-I_M + 2} \to 0
\end{aligned}$$

as $M \to \infty$.

Next we look at horizontal curves. Let γ be a C^1-curve in $\mathbb{R}^2$ so that

$$\left|\frac{\partial \gamma}{\partial x}\right| \geqslant \frac{|\gamma'|}{3}.$$

Take $i \in \mathbb{N}$ and $t \in \{0, \dots, M_i - 1\}$. Now there are at most two $k \in \{1, \dots, I_{i+1}\}$ so that

$$\gamma \cap S_{i+1, tI_{i+2} + k} \neq \emptyset.$$

By repeating this observation,

$$\mu(\gamma) \leqslant \prod_{i=2}^{M} \frac{2}{I_i} \to 0$$

as $M \to \infty$. Take any C^1-curve γ in $\mathbb{R}^2$. Because it can be covered with a countable collection of vertical and horizontal C^1-curves defined as above, we have $\mu(\gamma) = 0$. Thus, the measure μ is purely 1-unrectifiable.

Let $\ell \in G(2,1)$ be the horizontal line. We show that cones around ℓ have small measure in the sense of equality (5.9). To do this fix $0 < \alpha < 1$ and take the smallest $i_0 \in \{3, 4, \dots\}$ so that

$$\frac{1}{I_{i_0}} < \frac{\sqrt{1-\alpha}}{4}. \tag{5.12}$$

Now take $i \in \{i_0 + 1, i_0 + 2, \dots\}$, a point $x \in \operatorname{spt}(\mu)$ and a radius $r \in [M_i^{-1}, M_{i-1}^{-1}]$. Let $k_1 \in \mathbb{N}$ so that $x \in S_{i,k_1}$. Assume that there are at most two $k' \in \mathbb{N}$ so that

$$X(x, r, \ell, \alpha) \cap S_{i+1,k'} \neq \emptyset.$$

Then

$$\mu(X(x, r, \ell, \alpha)) \leqslant \frac{2\mu(B(x, r))}{I_{i+1}}. \tag{5.13}$$

Assume then that there are at least three such k'. If this is the case, then the cone $X(x, r, \ell, \alpha)$ must hit another large vertical strip S_{i,k_2} with $k_2 \in \{k_1 - 1, k_1 + 1\}$. Inequality (5.12) yields the existence of a block $B_{i,k_1,u} \subset B(x, r)$, whose vertical distance to the centre of the strip S_{i,k_1} is strictly less than the vertical distance from the centre of the strip S_{i,k_2} to any of the blocks $B_{i,k_2,u'}$ that intersect the cone $X(x, r, \ell, \alpha)$. From equation (5.10) we see that the measure is concentrated in the centre of the vertical strips and we get

$$\mu(B_{i,k_1,u}) \geqslant \frac{(2I_i)^u}{2\sum_{p=1}^{u-1}(2I_i)^p}\mu(X(x, r, \ell, \alpha))$$

yielding

$$\mu(X(x, r, \ell, \alpha)) \leqslant \frac{2\mu(B(x, r))}{I_i}.$$

This together with (5.13) proves (5.9) as i tends to ∞. □

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