NONSYMMETRIC CONICAL UPPER DENSITY AND k-POROSITY

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Abstract. We study how the Hausdorff measure is distributed in nonsymmetric narrow cones in $\mathbb{R}^n$. As an application, we find an upper bound close to $n-k$ for the Hausdorff dimension of sets with large k-porosity. With k-porous sets we mean sets which have holes in k different directions on every small scale.

1. Introduction

It is a well known fact that for a set $A \subset \mathbb{R}^n$ with finite s-dimensional Hausdorff measure, $\mathcal{H}^s(A) < \infty$, we have

$$1 \le \limsup_{r \downarrow 0} \frac{\mathcal{H}^s\big(A \cap B(x,r)\big)}{r^s} \le 2^s \tag{1.1}$$

for $\mathcal{H}^s$-almost every $x \in A$. For a proof, see, for example, [12, Theorem 6.2(1)]. This is analogous to the classical Lebesgue Density Theorem. Using this fact, we know roughly how much of A there is in small balls. Mattila [11] studied how A is distributed in such balls. He was able to estimate how much of A there is near $(n-m)$-planes. More precisely, assuming $0 \le m < s \le n$ and denoting

$$\begin{aligned} X(x,V,\alpha) &= \{y \in \mathbb{R}^n : \operatorname{dist}(y-x,V) < \alpha|y-x|\}, \\ X(x,r,V,\alpha) &= X(x,V,\alpha) \cap B(x,r), \end{aligned}$$

as $x \in \mathbb{R}^n$, $V \in G(n,m)$, $r > 0$, and $0 < \alpha \le 1$, he proved that there exists a constant $c = c(n,m,s,\alpha) > 0$ such that

$$\limsup_{r \downarrow 0} \inf_{V \in G(n,n-m)} \frac{\mathcal{H}^s\big(A \cap X(x,r,V,\alpha)\big)}{r^s} \ge c \tag{1.2}$$

for $\mathcal{H}^s$-almost every $x \in A$ whenever $A \subset \mathbb{R}^n$ is such that $\mathcal{H}^s(A) < \infty$. Here $G(n,m)$ denotes the collection of all m-dimensional linear subspaces of $\mathbb{R}^n$, see [12, §3.9]. Actually (1.2) is just a special case of Mattila's result, as his theorem can be applied also for more general cones, see [11, Theorem 3.3].

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In Theorem 2.5 we show that if A is as above, then it cannot be concentrated in too small regions, not even inside the cones $X(x,r,V,\alpha)$. More precisely, denoting

$$\begin{aligned} H(x,\theta) &= \{y \in \mathbb{R}^n : (y-x)\cdot\theta > 0\}, \\ H(x,\theta,\eta) &= \{y \in \mathbb{R}^n : (y-x)\cdot\theta > \eta|y-x|\}, \end{aligned}$$

for $x \in \mathbb{R}^n$, $\theta \in S^{n-1}$, and $0 < \eta \le 1$, we prove under the same assumptions as in (1.2) that there exists a constant $c = c(n,m,s,\alpha,\eta) > 0$ such that

$$\limsup_{r\downarrow 0} \inf_{\substack{\theta\in S^{n-1} \\ V\in G(n,n-m)}} \frac{\mathcal{H}^s\big(A \cap X(x,r,V,\alpha) \setminus H(x,\theta,\eta)\big)}{r^s} \ge c$$

for $\mathcal{H}^s$-almost every $x \in A$. Here S^{n-1} denotes the unit sphere of $\mathbb{R}^n$. To help the geometric visualization, it might be helpful to take α and η close to 0 and $\theta \in V \cap S^{n-1}$. Our method gives also a more elementary proof for (1.2) and it can also be used to obtain similar results for more general measures, see Theorem 2.7.

The nonsymmetric conical upper density theorem is essential in our application to k-porous sets, that is, the sets with $\mathrm{por}_k > 0$, see (1.5). The notation of porosity, or 1-porosity using our terminology, has arisen from the study of dimensional estimates related, for example, to the boundary behavior of quasiconformal mappings. See Koskela and Rohde [9], Martio and Vuorinen [10], Sarvas [15], Trocenko [17], and Väisälä [18]. The dimensional properties of 1-porous sets are well known. Using a version of (1.2), Mattila showed that if porosity is close to its maximum value $\frac{1}{2}$, then the dimension cannot be much bigger than $n-1$. More precisely,

$$\sup\{s > 0 : \mathrm{por}_1(A) > \varrho \text{ and } \dim_{\mathrm{H}}(A) > s \text{ for some } A \subset \mathbb{R}^n\} \longrightarrow n-1 \tag{1.3}$$

as $\varrho \to \frac{1}{2}$. Here $\dim_{\mathrm{H}}$ refers to the Hausdorff dimension. Later Salli [14] generalized this result for the Minkowski dimension, and found the correct asymptotics. The concept of 1-porosity has also been generalized for measures, and it leads to similar kind of dimension bounds. See Järvenpää and Järvenpää [4] and references therein.

Motivated by the fact that each $V \in G(n,n-1)$ has maximal 1-porosity, we introduce a porosity condition which describes also sets whose dimension is smaller than $n-1$. For any integer $0 < k \le n$, $x \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$, and $r > 0$ we set

$$\begin{aligned} \mathrm{por}_k(A,x,r) = \sup\{\varrho :\ & \text{there are } z_1,\ldots,z_k \in \mathbb{R}^n \text{ such that} \\ & B(z_i,\varrho r) \subset B(x,r) \setminus A \text{ for every } i, \\ & \text{and } (z_i-x)\cdot(z_j-x) = 0 \text{ for } i \ne j\}. \end{aligned} \tag{1.4}$$

Here $\cdot$ is the inner product. The *k-porosity of A at a point x* is defined to be

$$\mathrm{por}_k(A,x) = \liminf_{r\downarrow 0} \mathrm{por}_k(A,x,r),$$

and the *k-porosity of A* is given by

$$\mathrm{por}_k(A) = \inf_{x\in A} \mathrm{por}_k(A,x). \tag{1.5}$$

This means that k-porous sets have holes in k orthogonal directions near each of its points in every small scale. We shall now give a concrete example where k-porosity occurs naturally. Suppose $0 < \lambda < \frac{1}{2}$ and let $C_\lambda \subset \mathbb{R}$ be the usual λ-Cantor set, see [12, §4.10]. It is clearly a 1-porous set with $\mathrm{por}_1(C_\lambda) \approx \frac{1}{2} - \lambda$. Mattila's result (1.3) implies that $\dim_{\mathrm{H}}(C_\lambda) \to 0$ as $\mathrm{por}_1(C_\lambda) \to \frac{1}{2}$. Of course, we could obtain the same information just by calculating the Hausdorff dimension of the self-similar set C_λ and letting $\lambda \to 0$, but our aim was to provide the reader with an illustrative example. The sets $C_\lambda \times C_\lambda \subset \mathbb{R}^2$ and $C_\lambda \times C_\lambda \times [0,1] \subset \mathbb{R}^3$ are clearly 2-porous with $\mathrm{por}_2 \approx \frac{1}{2} - \lambda$. For these sets (1.3) does not give any reasonable dimension bound. However, it would be desirable to see, also in terms of porosity, that $\dim_{\mathrm{H}}(C_\lambda \times C_\lambda) \to 0$ and $\dim_{\mathrm{H}}(C_\lambda \times C_\lambda \times [0,1]) \to 1$ as $\lambda \to 0$. This follows as an immediate application of Theorem 3.2. Using our nonsymmetric conical upper density theorem, we show that

$$\sup\{s > 0 : \mathrm{por}_k(A) > \varrho \text{ and } \dim_{\mathrm{H}}(A) > s \text{ for some } A \subset \mathbb{R}^n\} \longrightarrow n - k$$

as $\varrho \to \frac{1}{2}$. Observe also that in the proof of Theorem 3.2 the orthogonality in (1.4) plays no rôle and we may replace it by an assumption of a uniform lower bound for the angles between $z_i - x$ and the $(k-1)$-plane spanned by vectors $z_j - x$, $i \neq j$.

Let us now discuss the situation when porosity is small. It is well known (for example, see [10]) that if $A \subset \mathbb{R}^n$ with $\mathrm{por}_1(A, x, r) \geq \varrho > 0$ for all $x \in A$ and $0 < r < r_0$, then

$$\dim_{\mathrm{M}}(A) < n - c\varrho^n, \tag{1.6}$$

where $c > 0$ depends only on n, and $\dim_{\mathrm{M}}$ refers to the Minkowski dimension, see [12, §5.3]. It might be possible to get a better estimate if por_1 is replaced by por_k for some $k > 1$, but this condition does not feel very natural if the size of the holes is small. However, if $V \in G(n,m)$ is fixed and the condition $\mathrm{por}_1(A, x, r) \geq \varrho$ is replaced by

$$\sup\bigl\{\varrho' : B(z, \varrho' r) \subset B(x,r) \setminus A \text{ for some } z \in V + \{x\}\bigr\} \geq \varrho,$$

then n in (1.6) can be replaced by m, see Theorem 4.3. This is a rather immediate consequence of (1.6), but our main point is to give a simple proof for (1.6) using iterated function systems.

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2. Nonsymmetric conical upper density

We shall first prove a density theorem for nonsymmetric regions and then prove our main theorem by using a similar argument on $(n-m)$-planes. The proofs rely on the following geometric fact.

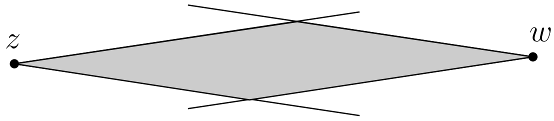

FIGURE A. All points lying on the gray region form a large angle with points z and w.

Lemma 2.1. *For given $0 < \beta < \pi$, there is $q = q(n, \beta) \in \mathbb{N}$ such that in any set of q points in $\mathbb{R}^n$, there are always three points which determine an angle between β and π.*

Remark 2.2. Erdős and Füredi [1] have shown that for the smallest possible choice of q it holds that

$$2^{(\pi/(\pi-\beta))^{n-1}} \leq q(n, \beta) \leq 2^{(4\pi/(\pi-\beta))^{n-1}} + 1.$$

For the convenience of the reader we shall give below a different proof which establishes the existence of some such q. The estimate that we get here for q is, however, quite bad compared to the best possible one.

Proof. Let A be a set of points in $\mathbb{R}^n$ so that all angles formed by its points are less than β. Let us fix $0 < \eta < 1$ and cover $\mathbb{R}^n \setminus \{0\}$ by cones $C_i = H(0, \theta_i, \eta)$, $i \in \{1, 2, \ldots, k\}$, where the constant $k = k(n, \eta) \in \mathbb{N}$ depends only on n and η. To visualize the situation, note that if β is close to π, then η is close to 1 and cones C_i are very narrow. To simplify the notation, we denote $C_{i,y} = C_i + \{y\}$ for $y \in \mathbb{R}^n$.

For any index $i_1 i_2 \cdots i_j$, where $j \in \mathbb{N}$ and $i_m \in \{1, 2, \ldots, k\}$ for $1 \leq m \leq j$, we define sets $A_{i_1 i_2 \cdots i_j}$ in the following way: We begin by fixing $x \in A$ and setting $A_i = A \cap C_{i,x}$ for $1 \leq i \leq k$. If $A_{i_1 i_2 \cdots i_j}$ has been defined, we choose $y \in A_{i_1 i_2 \cdots i_j}$ and define $A_{i_1 i_2 \cdots i_j l} = A_{i_1 i_2 \cdots i_j} \cap C_{l,y}$ for $1 \leq l \leq k$ (if $A_{i_1 i_2 \cdots i_j}$ is empty, then so is $A_{i_1 i_2 \cdots i_j l}$). We refer to y as the corner of $A_{i_1 i_2 \cdots i_j l}$. It follows directly from the definition of the sets $A_{i_1 i_2 \cdots i_j}$ that

$$\operatorname{card} A_{i_1 i_2 \cdots i_j} \leq 1 + \sum_{l=1}^{k} \operatorname{card} A_{i_1 i_2 \cdots i_j l}.$$

Iterating this, we get

$$\operatorname{card} A \leq \sum_{j=0}^{k} k^j + \sum_{i_1 i_2 \cdots i_k} \sum_{l=1}^{k} \operatorname{card} A_{i_1 i_2 \cdots i_k l}. \tag{2.1}$$

The main point of the proof is the observation that if $\eta = \eta(\beta)$ is chosen to be close enough to 1 in the beginning, then the following is true: If z and w are the corners of $A_{i_1 i_2 \cdots i_j}$ and $A_{i_1 i_2 \cdots i_j i_{j+1} \cdots i_m}$, respectively, and if $z \in C_{i_m, w}$, then $A \cap C_{i_j, z} \cap C_{i_m, w} = \emptyset$. See Figure A. It follows by induction from the above fact

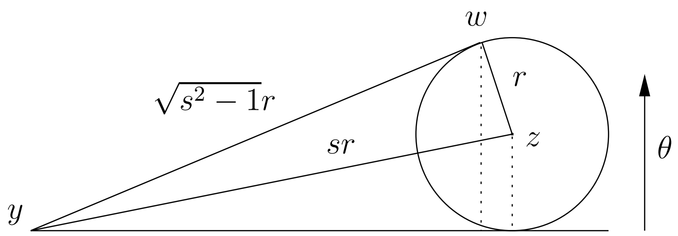

FIGURE B. Illustration for the proof of Lemma 2.3.

that for given $A_{i_1 i_2 \cdots i_j}$ we have

$$\operatorname{card}\{l : A_{i_1 i_2 \cdots i_j l} \neq \emptyset\} \leq k - j.$$

In particular, $A_{i_1 i_2 \cdots i_{k+1}} = \emptyset$ for any choice of $i_1 i_2 \ldots i_{k+1}$. Combined with (2.1), this gives $\operatorname{card} A \leq \sum_{j=0}^{k} k^j$. This number depends only on $k = k(n, \beta)$ and the claim follows. $\square$

For $0 < \eta \leq 1$ we define

$$t(\eta) = \sqrt{\frac{\eta^2 + 4}{\eta^2}},$$
$$\gamma(\eta) = \frac{1}{t(\eta)}.$$

Notice that $t(\eta) \geq 2$ and $\eta/\sqrt{5} \leq \gamma(\eta) \leq \eta/2$.

Lemma 2.3. *Suppose that $y \in \mathbb{R}^n$, $\theta \in S^{n-1}$, $0 < \eta \leq 1$, $t = t(\eta)$, and $\gamma = \gamma(\eta)$. If $z \in \mathbb{R}^n \setminus \big(B(y, tr) \cup H(y, \theta, \gamma)\big)$, then*

$$B(z, r) \cap H(y, \theta, \eta) = \emptyset.$$

Proof. Take $w \in \mathbb{R}^n$ such that it maximizes $(w - y) \cdot \theta / |w - y|$ in the closure of $B(z, r)$. It suffices to prove that $(w - y) \cdot \theta / |w - y| < \eta$, see Figure B. It is straightforward to check that $\eta\sqrt{s^2 - 1} \geq 1 + \gamma s$ when $s \geq t$. Denoting now $s = |y - z|/r$, we have $s \geq t > 1$ and thus

$$\begin{aligned}(w - y) \cdot \theta &< r + \gamma|y - z| = (1 + \gamma s)r \\ &\leq \eta\sqrt{s^2 - 1}r = \eta|w - y|,\end{aligned}$$

which finishes the proof. $\square$

Theorem 2.4. *Suppose $0 < \eta \leq 1$ and $0 < s \leq n$. Then there is a constant $c = c(n, s, \eta) > 0$ such that*

$$\limsup_{r \downarrow 0} \inf_{\theta \in S^{n-1}} \frac{\mathcal{H}^s\big(A \cap B(x, r) \setminus H(x, \theta, \eta)\big)}{r^s} \geq c$$

for $\mathcal{H}^s$ almost every $x \in A$ whenever $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$.

Proof. Take $c > 0$ and assume that there exists a Borel set $B \subset \mathbb{R}^n$ with $\mathcal{H}^s(B) > 0$ such that for each $x \in B$ and $0 < r < r_0$ there is $\theta \in S^{n-1}$ for which

$$\mathcal{H}^s\big(B \cap B(x,r) \setminus H(x,\theta,\eta)\big) < cr^s. \tag{2.2}$$

It suffices to find a positive lower bound for c in terms of n, s, and η.

Using (1.1), and replacing B by a suitable subset if necessary, we may assume that

$$\mathcal{H}^s\big(B \cap B(x,r)\big) < 2^{s+1}r^s \tag{2.3}$$

for all $0 < r < r_0$ and $x \in B$. Moreover, using the lower estimate of (1.1), we find $0 < r < r_0/3$ and $x \in B$ such that

$$\mathcal{H}^s\big(B \cap B(x,r)\big) > \tfrac{1}{2}r^s. \tag{2.4}$$

Set $t = t(\eta)$, $\gamma = \gamma(\eta)$, and take $0 < \delta < 1$. Let us fix $\beta < \pi$ such that the opening angle of $H(x,\theta,\gamma)$ is smaller than β, and let $q = q(n,\beta)$ be as in Lemma 2.1. We may cover the set $B \cap B(x,r)$ by $4^n\delta^{-n}$ balls of radius δr with centers in B. Using (2.4), we notice that there exists $x_1 \in B \cap B(x,r)$ such that

$$\mathcal{H}^s\big(B \cap B(x_1,\delta r)\big) > 4^{-n}\delta^n 2^{-1} r^s.$$

The set $B \cap B(x,r) \setminus B(x_1,t\delta r)$ can also be covered by $4^n\delta^{-n}$ balls of radius δr with centers in B. Whence, using (2.3) and (2.4),

$$\mathcal{H}^s\big(B \cap B(x,r) \setminus B(x_1,t\delta r)\big) > (\tfrac{1}{2} - 2^{s+1}t^s\delta^s)r^s.$$

If $\frac{1}{2} - 2^{s+1}t^s\delta^s > 0$, we find $x_2 \in B \cap B(x,r) \setminus B(x_1,t\delta r)$ for which

$$\mathcal{H}^s\big(B \cap B(x_2,\delta r)\big) > 4^{-n}\delta^n(\tfrac{1}{2} - 2^{s+1}t^s\delta^s)r^s.$$

Choosing $\delta = \delta(n,s,\eta) > 0$ small enough and continuing in this manner, we find q points $x_1,\ldots,x_q \in B \cap B(x,r)$ with $|x_i - x_j| \ge t\delta r$ for $i \ne j$, such that for each $i \in \{1,\ldots,q\}$ we have

$$\begin{aligned}\mathcal{H}^s\big(B \cap B(x_i,\delta r)\big) &> 4^{-n}\delta^n\big(\tfrac{1}{2} - (q-1)2^{s+1}t^s\delta^s\big)r^s \\ &=: c(n,s,\eta)(3r)^s,\end{aligned} \tag{2.5}$$

where $c(n,s,\eta) > 0$.

According to Lemma 2.1, we may choose three points y, y_1, y_2 from the set $\{x_1,\ldots,x_q\}$ such that for each $\theta \in S^{n-1}$ there is $i \in \{1,2\}$ for which $y_i \in \mathbb{R}^n \setminus \big(B(y,t\delta r) \cup H(y,\theta,\gamma)\big)$. We obtain, using Lemma 2.3, that for each $\theta \in S^{n-1}$ there is $i \in \{1,2\}$ such that

$$B(y_i,\delta r) \subset B\big(y, 2(1+\delta)r\big) \setminus H(y,\theta,\eta).$$

Thus, applying (2.5), we have

$$\mathcal{H}^s\big(B \cap B(y,3r) \setminus H(y,\theta,\eta)\big) > c(n,s,\eta)(3r)^s$$

for all $\theta \in S^{n-1}$. Recalling (2.2), we conclude that $c \ge c(n,s,\eta)$. The proof is finished. $\square$

Theorem 2.5. *Suppose* $0 < \alpha, \eta \leq 1$ *and* $0 \leq m < s \leq n$. *Then there is a constant* $c = c(n, m, s, \alpha, \eta) > 0$ *such that*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n,n-m)}} \frac{\mathcal{H}^s\big(A \cap X(x, r, V, \alpha) \setminus H(x, \theta, \eta)\big)}{r^s} \geq c$$

for $\mathcal{H}^s$ *almost every* $x \in A$ *whenever* $A \subset \mathbb{R}^n$ *with* $\mathcal{H}^s(A) < \infty$.

Proof. For any $V, W \in G(n, n-m)$, we set $d(V, W) = \sup_{x \in V \cap S^{n-1}} \operatorname{dist}(x, W)$. With this metric $G(n, n-m)$ is a compact metric space, see Salli [13]. Defining for each $V \in G(n, n-m)$ a set $\{W : d(V, W) < \alpha/2\}$ we notice that a finite number of these sets is still a cover. We assume that the sets assigned to the planes $V_1, \ldots, V_l$, where $l = l(n, m, \alpha)$, cover $G(n, n-m)$. For any W, it holds that $d(V_i, W) < \alpha/2$ with some $i \in \{1, \ldots, l\}$. This implies $X(0, V_i, \alpha/2) \subset X(0, W, \alpha)$. Thus, for each $W \in G(n, n-m)$, there is i such that

$$X(x, r, W, \alpha) \supset X(x, r, V_i, \alpha/2) \tag{2.6}$$

for all $r > 0$ and $x \in \mathbb{R}^n$. We shall prove that if $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$, then

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ i \in \{1,\ldots,l\}}} \frac{\mathcal{H}^s\big(A \cap X(x, r, V_i, \alpha/2) \setminus H(x, \theta, \eta)\big)}{r^s} \geq c(n, m, s, \alpha, \eta)$$

for $\mathcal{H}^s$ almost every $x \in A$ from which the claim follows easily by using (2.6).

Take $c > 0$ and assume that there is a Borel set $B \subset \mathbb{R}^n$ with $\mathcal{H}^s(B) > 0$ such that for each $x \in B$ and $0 < r < r_0$ there are i and $\theta \in S^{n-1}$ for which

$$\mathcal{H}^s\big(B \cap X(x, r, V_i, \alpha/2) \setminus H(x, \theta, \eta)\big) < cr^s.$$

According to (1.1) we may assume that

$$\mathcal{H}^s\big(B \cap B(x, r)\big) < 2^{s+1} r^s \tag{2.7}$$

for all $0 < r < r_0$ and $x \in B$. Using the lower estimate of (1.1), we find $0 < r < r_0/3$ and $x \in B$ such that

$$\mathcal{H}^s\big(B \cap B(x, r)\big) > \tfrac{1}{2} r^s. \tag{2.8}$$

Next we define

$$\begin{aligned} B_i = \big\{z \in B : \mathcal{H}^s\big(B \cap X(z, 3r, V_i, \alpha/2) \setminus H(z, \theta, \eta)\big) < c(3r)^s \\ \text{for some } \theta \in S^{n-1}\big\}. \end{aligned} \tag{2.9}$$

Since $\bigcup_{i=1}^l B_i = B$, we infer from (2.8) that there is $i_0 \in \{1, \ldots, l\}$ for which

$$\mathcal{H}^s\big(B_{i_0} \cap B(x, r)\big) > 2^{-1} l^{-1} r^s.$$

Let $t = \max\{5/\alpha, t(\eta)\}$, choose $q = (n, \eta)$ as in the proof of Theorem 2.4, and define $0 < \varepsilon < 1$ so that

$$4^{-m} 2^{-1} l^{-1} \varepsilon^m - (q-1) 2^{s+1} t^s \varepsilon^s = 4^{-m-1} l^{-1} \varepsilon^m\,, \tag{2.10}$$

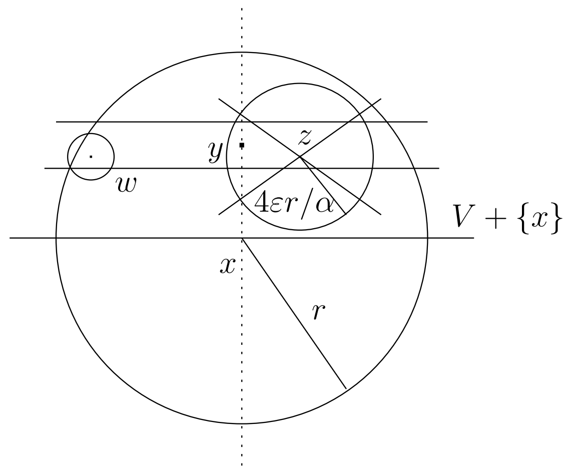

FIGURE C. Illustration for the proof of Theorem 2.5.

recall that $s>m$ so that this is possible. Since the set $(V_{i_0}^{\perp}+\{x\})\cap B(x,r)$ may be covered by $4^m\varepsilon^{-m}$ balls of radius εr, there exists $y\in(V_{i_0}^{\perp}+\{x\})\cap B(x,r)$ such that

$$\mathcal{H}^s\big(B_{i_0}\cap B(x,r)\cap P_{V_{i_0}^{\perp}}^{-1}(B(y,\varepsilon r))\big) > 4^{-m}2^{-1}l^{-1}\varepsilon^m r^s. \tag{2.11}$$

We now argue as in the proof of Theorem 2.4 above. We first observe that the slice $S=B_{i_0}\cap B(x,r)\cap P_{V_{i_0}^{\perp}}^{-1}\big(B(y,\varepsilon r)\big)$ may be covered by $c_1^{-1}\varepsilon^{m-n}$ balls of radius εr for a constant $c_1=c_1(n,m)>0$. Then we use (2.11), (2.7), and (2.10) to find points $\{x_1,\ldots,x_q\}\in S$ such that $|x_i-x_j|\ge t\varepsilon r$ whenever $i\ne j$ and

$$\begin{aligned}\mathcal{H}^s\big(S\cap B(x_i,\varepsilon r)\big) &> c_1\varepsilon^{n-m}\left(4^{-m}2^{-1}l^{-1}\varepsilon^m r^s-(q-1)2^{s+1}t^s\varepsilon^s r^s\right)\\ &= c_2(3r)^s\end{aligned} \tag{2.12}$$

for all i. Here $c_2=c_2(n,m,s,\alpha,\eta)=c_1 3^{-s}4^{-m-1}l^{-1}\varepsilon^m$. Now the same geometric argument as in the proof of Theorem 2.4 implies that there is a point $z\in\{x_1,\ldots,x_q\}$ such that for each $\theta\in S^{n-1}$ we may find $w\in\{x_0,\ldots,x_q\}\setminus\{z\}$ so that

$$B(w,\varepsilon r)\subset B\big(z,(2+\varepsilon)r\big)\setminus\big(H(z,\theta,\eta)\cap B(z,4\varepsilon r/\alpha)\big).$$

Since also

$$P_{V_{i_0}^{\perp}}^{-1}\big(B(y,\varepsilon r)\big)\cap B(z,3r)\setminus B(z,4\varepsilon r/\alpha)\subset X(z,3r,V_{i_0},\alpha/2),$$

see Figure C, we get

$$\inf_{\theta\in S^{n-1}}\mathcal{H}^s\big(B\cap X(z,3r,V_{i_0},\alpha/2)\setminus H(z,\theta,\eta)\big)\ge c_2(3r)^s.$$

by (2.12). Now $z\in B_{i_0}$ and we conclude, using (2.9), that $c\ge c_2=c_2(n,m,s,\alpha,\eta)$. This completes the proof. $\square$

Remark 2.6. Inspecting the proofs, one can read explicit expressions for the constants in Theorems 2.4 and 2.5. In Theorem 2.4, one gets $c \geq 2^{c_1/(-s\eta^{n-1})}$ and in Theorem 2.5, one obtains $c \geq \alpha^{c_3/(s-m)}\, 2^{c_2/((m-s)\eta^{n-1})}$. The constants $0 < c_1, c_2, c_3 < \infty$ here depend only on n. The estimates obtained in this way are probably rather far from being optimal, although the best values are not known.

Our method can be applied also in a more general setting. A similar proof as above gives the following result. If μ is a measure on $\mathbb{R}^n$, $h\colon (0, r_0) \to (0, \infty)$, and $x \in \mathbb{R}^n$, we define $\overline{D}(\mu, x)$ and $\underline{D}(\mu, x)$ as the lower and upper limits, respectively, of the ratio $\mu\big(B(x, r)\big)/h(r)$ as $r \downarrow 0$.

Theorem 2.7. *Suppose* $0 \leq m < n$ *and* $h\colon (0, r_0) \to (0, \infty)$ *is a function with*

$$\frac{h(\varepsilon r)}{\varepsilon^m h(r)} \longrightarrow 0 \qquad \text{uniformly for all } 0 < r < r_0 \tag{2.13}$$

as $\varepsilon \downarrow 0$. *Let* μ *be a measure on* $\mathbb{R}^n$ *with* $\overline{D}(\mu, x) < \infty$ *for* μ*-almost all* $x \in \mathbb{R}^n$. *For every* $0 < \alpha, \eta \leq 1$, *there is a constant* $c = c(n, m, h, \alpha, \eta) > 0$ *such that*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mu\big(X(x, r, V, \alpha) \setminus H(x, \theta, \eta)\big)}{h(r)} \geq c\overline{D}(\mu, x)$$

for μ*-almost every* $x \in \mathbb{R}^n$.

Let us make few comments related to the above theorem. Suppose that h fulfills condition (2.13). Let $\mathcal{H}_h$ be the generalized Hausdorff measure which is constructed using h as a gauge function, see [12, §4.9]. If $\mu = \mathcal{H}_h|_A$, where $\mathcal{H}_h(A) < \infty$, then $\overline{D}(\mu, x) < \infty$ for μ-almost every $x \in \mathbb{R}^n$, and thus Theorem 2.7 can be applied.

There are many natural gauge functions, such as $h(r) = r^s \log(1/r)$ where $m < s < n$, which satisfy (2.13). However, some interesting cases, such as $h(r) = r^m/\log(1/r)$, are not covered by this condition.

It seems to be unknown whether a similar result as Theorem 2.7 holds if one replaces the condition $\overline{D}(\mu, x) < \infty$ by $\underline{D}(\mu, x) < \infty$. The most interesting example falling into this category is obtained when $\mu = \mathcal{P}^s|_A$ and $h(r) = r^s$, where $\mathcal{P}^s(A) < \infty$ and $m < s < n$. Here $\mathcal{P}^s$ denotes the s-dimensional packing measure, see [12, §5.10]. See also Suomala [16] for related theorems.

3. Sets with large k-porosity

Mattila [11] proved Theorem 2.5 in the case $m = n - 1$. Using this, he obtained the desired dimension bounds for 1-porous sets, see (1.3). Our result for k-porous sets follows applying a similar argument.

For $\sqrt{2}-1 < \varrho < \frac{1}{2}$ we define

$$t(\varrho) = \frac{1}{\sqrt{1-2\varrho}}\,,$$
$$\delta(\varrho) = \frac{1-\varrho-\sqrt{\varrho^2+2\varrho-1}}{\sqrt{1-2\varrho}}\,.$$

Notice that $\delta(\varrho) \to 0$ as $\varrho \to \frac{1}{2}$.

Lemma 3.1. *Suppose $x \in \mathbb{R}^n$, $r > 0$, $\sqrt{2}-1 < \varrho < \frac{1}{2}$, $t = t(\varrho)$, and $\delta = \delta(\varrho)$. If $z \in \mathbb{R}^n \setminus \{x\}$ is such that $B(z, \varrho t r) \subset B(x, tr)$, then*

$$H(x + \delta r\theta, \theta) \cap B(x, r) \subset B(z, \varrho t r),$$

where $\theta = (z-x)/|z-x|$.

Proof. To simplify the notation, we assume $r = 1$, $x = 0$, and $\theta = e_1 = (1, 0, \ldots, 0)$. This will not affect the generality. Let $y \in B(0,1) \setminus B(z, \varrho t)$. We have to show that

$$y \notin H(x + \delta\theta, \theta). \tag{3.1}$$

By the Pythagorean Theorem we have

$$|z - y_1| = \sqrt{|z-y|^2 - |y-y_1|^2} \geq \sqrt{(\varrho t)^2 - 1}.$$

Using this, we obtain

$$y_1 = |z| - |z - y_1| \leq t - \varrho t - \sqrt{(\varrho t)^2 - 1} = \delta,$$

which implies (3.1). □

Theorem 3.2. *Suppose $0 < k \leq n$. Then*

$$\sup\{s > 0 : \mathrm{por}_k(A) > \varrho \text{ and } \dim_{\mathrm{H}}(A) > s \text{ for some } A \subset \mathbb{R}^n\} \longrightarrow n - k$$

as $\varrho \to \frac{1}{2}$.

Proof. Assume on the contrary that there exists $s > n-k$ such that for each $\sqrt{2}-1 < \varrho < \frac{1}{2}$ there is a set A_ϱ for which $\dim_{\mathrm{H}}(A_\varrho) > s$ and $\mathrm{por}_k(A_\varrho) > \varrho$. Take $\sqrt{2}-1 < \varrho < \frac{1}{2}$ and such a set A_ϱ. Now A_ϱ has a subset B for which $\dim_{\mathrm{H}}(B) > s$ and $\mathrm{por}_k(B, x, r) > \varrho$ for all $x \in B$ and $0 < r < r_0$ with some $r_0 > 0$. Clearly also the closure of B satisfies these conditions. Thus there is a closed set $F \subset \overline{B}$ (for example, use [2, Theorem 5.4]) such that $0 < \mathcal{H}^s(F) < \infty$ and

$$\mathrm{por}_k(F, x, r) > \varrho \qquad \text{for all } x \in F \text{ and } 0 < r < r_0.$$

Therefore, for any $x \in F$ and $0 < r < r_0/t$, there are $z_1, \ldots, z_k \in \mathbb{R}^n$ such that $B(z_i, \varrho t r) \subset B(x, tr) \setminus F$ for $i = 1, \ldots, k$, and $(z_i - x) \cdot (z_j - x) = 0$ for $i \neq j$. Put $\theta_i = (z_i - x)/|z_i - x|$. Applying now Lemma 3.1 we have $H(x + \delta r\theta_i, \theta_i) \cap B(x, r) \subset$

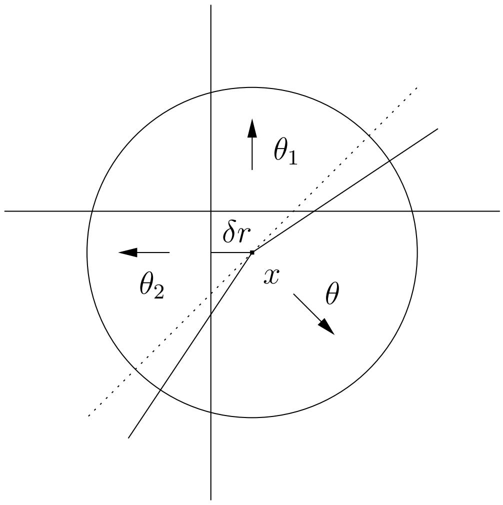

FIGURE D. Illustration for the proof of Theorem 3.2: the situation when $n = 2$ and $k = 2$.

$B(z_i, \varrho t r)$ for every i. Here $t = t(\varrho)$ and $\delta = \delta(\varrho)$. Thus

$$F \cap B(x, r) \subset \bigcap_{i=1}^{k} B(x, r) \setminus H(x + \delta r \theta_i, \theta_i). \tag{3.2}$$

Put $\theta = -\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \theta_i$ and take $V \in G(n, k)$ such that $\theta_i \in V$ for every i. Now choosing α and η small enough, we have, using (3.2), that

$$F \cap X(x, r, V, \alpha) \setminus H(x, \theta, \eta) \subset B(x, 2n^{1/2} \delta r). \tag{3.3}$$

Observe that the choice of α and η does not depend on δ and hence not on ϱ either. Figure D illustrates the situation. Using Theorem 2.5, we may fix $x \in F$ and $0 < r < r_0/t$ for which

$$\mathcal{H}^s\big(F \cap X(x, r, V, \alpha) \setminus H(x, \theta, \eta)\big) \geq c 2^{2s+1} n^{s/2} r^s, \tag{3.4}$$

where $c = c(n, k, s, \alpha, \eta) > 0$. By (1.1) we may assume that also

$$\mathcal{H}^s\big(F \cap B(x, 2n^{1/2} \delta r)\big) \leq 2^{2s+1} n^{s/2} \delta^s r^s. \tag{3.5}$$

Combining (3.3)–(3.5), we have $c 2^{2s+1} n^{s/2} r^s \leq 2^{2s+1} n^{s/2} \delta^s r^s$ and hence

$$s \leq \frac{\log c}{\log \delta(\varrho)}.$$

But the constant c does not depend on ϱ, and thus $\log c / \log \delta(\varrho) \to 0$ as $\varrho \to \frac{1}{2}$ giving a contradiction. $\square$

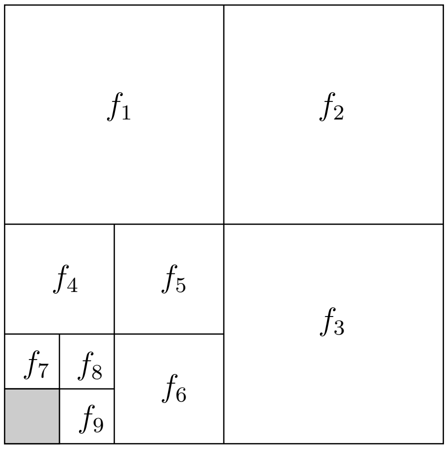

FIGURE E. Similitudes f_k in the proof Theorem 4.1 when $n = 2$ and $l = 3$.

4. SETS WITH SMALL POROSITY

Finally, let us briefly discuss the situation when porosity is small. The proof of the following theorem can be found for example in Martio and Vuorinen [10]. We shall give here a different proof, and then show how the theorem can be improved when more information on the location of the holes is given.

Theorem 4.1. *Let $A \subset \mathbb{R}^n$ be bounded and suppose that* $\mathrm{por}_1(A, x, r) \geq \varrho$ *for all $x \in A$ and $0 < r < r_0$. Then* $\dim_{\mathrm{M}}(A) < n - c\varrho^n$, *where $c > 0$ depends only on n.*

Proof. We may assume that $r_0 = 1$ and $A \subset [0, 1]^n$. Let us denote by $\mathcal{Q}_j$ the collection of all closed dyadic cubes $Q \subset [0, 1]^n$ with side length 2^{-j}. Let l be the smallest integer with $2^{-l+2} < \varrho/\sqrt{n}$. It is easy to see that for any $Q \in \mathcal{Q}_j$ there is $Q' \in \mathcal{Q}_{j+l}$ such that $Q' \subset Q$ and $Q' \cap A = \emptyset$. Let us fix one such Q' for each $Q \in \bigcup_{j=1}^{\infty} \mathcal{Q}_j$. Next we define a set $B \subset [0, 1]^n$ by setting

$$B = [0, 1]^n \setminus \bigcup_{j=0}^{\infty} \bigcup_{Q \in \mathcal{Q}_j} Q'. \tag{4.1}$$

For any $Q \in \mathcal{Q}_j$, let x_Q be the corner of Q which is nearest to the origin, and let $\widetilde{Q} = \{x_Q\} + [0, 2^{-j-1}]^n$. If we define $E \subset [0, 1]^n$ by setting

$$E = [0, 1]^n \setminus \bigcup_{j=0}^{\infty} \bigcup_{Q \in \mathcal{Q}_j} \mathrm{int}\widetilde{Q},$$

where int denotes the interior of a given set, then obviously $\dim_{\mathrm{M}}(E) \geq \dim_{\mathrm{M}}(B)$, see also [7]. The set E is the limit set of the iterated function system defined by the similitudes f_k, $k \in \{1, 2, \ldots, l(2^n - 1)\}$, see Figure E. For any $i \in \{1, \ldots, l\}$, there are 2^n-1 similitudes among $\{f_k\}_{k=1}^{l(2^n-1)}$ with contraction ratio 2^{-i}. Since the open set condition is clearly satisfied, the dimension $s=\dim_{\mathrm{M}}(E)=\dim_{\mathrm{H}}(E)$ is given by

$$(2^n-1)\sum_{i=1}^{l}2^{-is}=1, \tag{4.2}$$

see Hutchinson [3, §5]. This reduces to

$$2^{n-s}=1+(2^n-1)2^{-(l+1)s}$$

and since $\log_2(1+x)\geq x/\big((1+x)\log 2\big)$ for $x\geq 0$, we have

$$\begin{aligned}s&=n-\log_2\big(1+(2^n-1)2^{-(l+1)s}\big)\\&\leq n-\log_2\big(1+(1-2^{-n})2^{-ln}\big)\\&\leq n-\frac{2}{5\log 2}2^{-ln}\leq n-c\varrho^n,\end{aligned}$$

where $c=\big(2/(5\log 2)\big)2^{-3n}n^{-n/2}$. Because $A\subset B$ and $\dim_{\mathrm{M}}(B)\leq\dim_{\mathrm{M}}(E)=s$, we conclude that also $\dim_{\mathrm{M}}(A)\leq n-c\varrho^n$. □

In the above proof, the use of the self-similar set E is not a necessity, but it concretizes the situation. The key point in the proof is that for any cube $Q\subset\mathbb{R}^n$ which is small enough, one can find subcubes $Q_1,\ldots,Q_{l(2^n-1)}\subset Q$ such that $A\cap Q\subset\bigcup_{i=1}^{l(2^n-1)}Q_i$ and $\sum_{i=1}^{l(2^n-1)}\operatorname{diam}(Q_i)^s=\operatorname{diam}(Q)^s$, where s is given by (4.2). From this the desired dimension bound follows easily.

Remark 4.2. In a sense the above result is the best possible one. There is a constant $c'=c'(n)>0$ and sets A_ϱ, $0<\varrho<1/2$, with $\dim_{\mathrm{H}}(A_\varrho)>n-c'\varrho^n$, and $\operatorname{por}_1(A_\varrho,x,r)\geq\varrho$ for all $r>0$ and $x\in\mathbb{R}^n$. See, for example, Koskela and Rohde [9], or estimate the Hausdorff dimension of the set E from below.

Theorem 4.3. *Let $A\subset\mathbb{R}^n$ be bounded and suppose that there is $V\in G(n,m)$ such that for all $x\in A$ and $0<r<r_0$ one has*

$$\sup\big\{\varrho' : B(z,\varrho' r)\subset B(x,r)\setminus A \text{ for some } z\in V+\{x\}\big\}\geq\varrho. \tag{4.3}$$

Then $\dim_{\mathrm{M}}(A)<n-c\varrho^m$, where $c>0$ depends only on n and m.

Proof. Without losing the generality we may assume that $V=\mathbb{R}^m=\{x\in\mathbb{R}^n : x_{m+1}=x_{m+2}=\ldots=x_n=0\}$, $r_0=\sqrt{n}$, and $A\subset[0,1]^n$. Let $\mathcal{Q}_j$ be, as before, the collection of all closed dyadic cubes $Q\subset[0,1]^n$ with side length 2^{-j}, and let $\widetilde{\mathcal{Q}}_j=\{P_V(Q):Q\in\mathcal{Q}_j\}$ and $\mathcal{Q}'_j=\{P_{V^\perp}(Q):Q\in\mathcal{Q}_j\}$. Here P_V is the orthogonal projection onto V. Furthermore, let l be the smallest integer with $2^{-l+2}<\varrho/\sqrt{n}$.

We define a set $E=E_{l,m}\subset V$ as in the proof of Theorem 4.1. For $j\in\mathbb{N}$ we let $a_j=a_{j,l,m}$ denote the minimum number of cubes from the collection $\widetilde{\mathcal{Q}}_j$ that are needed to cover E. The proof of Theorem 4.1 yields that

$$\lim_{j\to\infty} \frac{\log a_j}{\log(2^j)} \le m - c2^{-ml}, \tag{4.4}$$

where $c > \frac{1}{2}$ is an absolute constant.

It is straightforward to convince oneself of the following fact: If $\widetilde{Q} \in \widetilde{\mathcal{Q}}_j$ and $Q' \in \mathcal{Q}'_{j+l}$, then there is $Q \in \mathcal{Q}_{j+l}$ such that $P_{V^\perp}(Q) = Q'$, $P_V(Q) \subset \widetilde{Q}$, and $A \cap Q = \emptyset$. From this observation it follows that given $Q' \in \mathcal{Q}'_j$, only a_j cubes from the collection $\{Q \in \mathcal{Q}_j : P_{V^\perp}(Q) = Q'\}$ touch the set A. Thus only $2^{j(n-m)}a_j$ cubes from the collection $\mathcal{Q}_j$ are needed to cover A. Using (4.4), we calculate

$$\begin{aligned}\dim_{\mathrm{M}}(A) &\le \limsup_{j\downarrow 0} \frac{\log(2^{j(n-m)}a_j)}{\log(2^j)} = n - m + \limsup_{j\downarrow 0} \frac{\log a_j}{\log(2^j)} \\ &\le n - c2^{-ml} \le n - c2^{-3m}n^{-m/2}\varrho^m.\end{aligned}$$

The proof is finished. □

Remark 4.4. Suppose that $V \in G(n,m)$ is fixed and $A \subset \mathbb{R}^n$ is such that (4.3) holds for every $x \in A$ and $0 < r < r_x$, where $r_x > 0$ depends on the point x. It follows immediately from Theorem 4.3 that $\dim_{\mathrm{H}}(A) \le \dim_{\mathrm{p}}(A) \le n - c\varrho^m$, where c is as in Theorem 4.3 and $\dim_{\mathrm{p}}$ denotes the packing dimension, see [12, §5.9]. The above dimension estimates are also sharp. Consider, for example, sets of the form $E \times \mathbb{R}^{n-m}$, where $E \subset \mathbb{R}^m$ is as in the proof of Theorem 4.1.

Remark 4.5. After the submission of this article in May 2004, there has been considerable progress in the study of conical densities and porosities. Most notably, the question posed after Theorem 2.7 has been answered positively in [8]. For improvements of Theorems 3.2 and 4.1, see [6] and [5], respectively.

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