ONE-SIDED DENSITY THEOREMS FOR MEASURES ON THE REAL LINE

VILLE SUOMALA

1. Introduction

If μ is a measure on $\mathbb{R}$, $A \subset \mathbb{R}$, and h is a positive function defined on some interval $]0, r_0[$, we define the following densities:

$$\underline{D}_h(\mu, A, x) = \liminf_{r \downarrow 0} \mu\left(A \cap [x-r, x+r]\right)/h(r),$$

$$\overline{D}_h(\mu, A, x) = \limsup_{r \downarrow 0} \mu\left(A \cap [x-r, x+r]\right)/h(r).$$

We also use the notations $\underline{D}_h(\mu, x) = \underline{D}_h(\mu, \mathbb{R}, x)$ and $\overline{D}_h(\mu, x) = \overline{D}_h(\mu, \mathbb{R}, x)$.

In the following three cases it is more or less justifiable to say that the function h carries information about the measure μ:

(1) $0 < \underline{D}_h(\mu, x) < \infty$ for μ-almost all $x \in \mathbb{R}$.
(2) $0 < \overline{D}_h(\mu, x) < \infty$ for μ-almost all $x \in \mathbb{R}$.
(3) $\underline{D}_h(\mu, x) = 0$ and $\overline{D}_h(\mu, x) = \infty$ for μ-almost all $x \in \mathbb{R}$.

If one of these conditions holds only in a set of positive measure but not almost everywhere, then we may consider a suitable restriction of μ, see Lemma 1.2.

When $h(r) = r^s$ for some $0 < s < 1$ and μ is the restriction of the s-dimensional Hausdorff measure, $\mathcal{H}^s$, to a set A with $0 < \mathcal{H}^s(A) < \infty$, then $1 \leq \overline{D}_h(\mu, x) \leq 2^s$ for μ-almost all $x \in \mathbb{R}$. It is well known that in this case one can determine the values of the one-sided densities $\underline{D}_h(\mu, [x, \infty[, x)$, $\underline{D}_h(\mu,]-\infty, x], x)$, $\overline{D}_h(\mu, [x, \infty[, x)$, and $\overline{D}_h(\mu,]-\infty, x], x)$ almost everywhere. Below, the notation $\mu \llcorner A$ is used for the restriction measure given by $\mu \llcorner A(B) = \mu(B \cap A)$ for $B \subset \mathbb{R}$.

Theorem 1.1 (Besicovitch [1]). *Let $0 < s < 1$, $h(r) = r^s$, $A \subset \mathbb{R}$ with $0 < \mathcal{H}^s(A) < \infty$, and $\mu = \mathcal{H}^s \llcorner A$. Then*

$$\underline{D}_h(\mu, [x, \infty[, x) = \underline{D}_h(\mu,]-\infty, x], x) = 0 \tag{1.1}$$

and

$$\overline{D}_h(\mu, [x, \infty[, x) = \overline{D}_h(\mu,]-\infty, x], x) = 1 \tag{1.2}$$

for μ-almost all $x \in \mathbb{R}$.

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In higher dimensions there are no "right" and "left" but then one can study the limits $\underline{D}_h(\mu, C, x)$ and $\overline{D}_h(\mu, C, x)$ for different kind of cones $C = C_x$. For the Hausdorff measures these so called conical density properties have been studied by various authors [2], [9], [13], and [11]. See also the monographs [5], [4], and [12]. Most of the conical density results which can be found from the literature deal with the Hausdorff measures $\mathcal{H}^s$. In [16] the author studied conical density properties of more general measures on $\mathbb{R}^n$ the main applications being the generalised Hausdorff and packing measures which are constructed using a suitable gauge function, see [16]. The importance of the conical density theorems is based on the fact that they may be used to derive geometric information of the given measure from a given metric information. That is, the values of the measure on small balls reflect the distribution of the measure.

In higher dimensions, the proofs become often quite technical and it is not always easy to see the underlying ideas. Thus, it is sometimes useful to express these ideas by giving the proofs on the real line only. In Theorem 2.1 we give a one-dimensional proof for one of the main results of [16]. Our second result, Theorem 3.1, is an upper density theorem for measures on the real-line which was stated in [16] without a proof. Our results should also be considered as a generalisations of Theorem 1.1.

For other recent results related to conical density properties of measures see [8], [14], [6], [7] and the thesis [15].

We are going to make use of the following lemma which is a simple consequence of the basic differentiation properties of measures. See [16, Lemma 2.6].

Lemma 1.2. *Suppose that μ is a Borel measure on $\mathbb{R}$ so that $\mu([x-r, x+r]) < \infty$ for μ-almost all $x \in \mathbb{R}$ with some $r = r(x) > 0$, $A \subset \mathbb{R}$ is a Borel set, and h is a positive function defined on some interval $]0, r_0[$. Then $\underline{D}_h(\mu, A, x) = \underline{D}_h(\mu, x)$ and $\overline{D}_h(\mu, A, x) = \overline{D}_h(\mu, x)$ for μ-almost all $x \in A$.*

2. Lower densities

In the following theorem we give a simple proof for Theorem 2.2 of [16] on $\mathbb{R}$. Although the proof is easier in the one-dimensional case, the key idea is the same in all dimensions; Consider a suitable exceptional set F and fill the space outside F effectively by cones that touch F on their vertexes.

Theorem 2.1. *Let $h\colon]0, r_0[\to]0, \infty[$ be a function such that*

$$\lim_{r \downarrow 0} h(r) = 0, \tag{h1}$$

$$\lim_{r \downarrow 0} h(r)/r = \infty, \text{ and} \tag{h2}$$

$$h(r_1) + h(r_2) \geq h(r_1 + r_2) \text{ whenever } r_1 + r_2 \leq r_0. \tag{h3}$$

If μ is a measure on $\mathbb{R}$ with $\underline{D}_h(\mu, x) < \infty$ for μ-almost all $x \in \mathbb{R}$, then

$$\underline{D}_h(\mu, [x, \infty[, x) = \underline{D}_h(\mu,]-\infty, x], x) = 0$$

for μ-almost all $x \in \mathbb{R}$.

Proof. Due to symmetry it suffices to prove that $\underline{D}_h(\mu, [x, \infty[, x) = 0$ for μ-almost all $x \in \mathbb{R}$. We may assume that μ is Borel regular since there is a Borel regular measure ν which equals μ for Borel sets. We may also assume that μ is finite since μ-almost all of $\mathbb{R}$ is contained in a countable collection of open intervals, each of finite μ-measure. Let $\alpha, \beta > 0$ and define

$$F = \{x \in \mathbb{R} \,:\, \mu([x, x + r]) \geq \alpha\, h(r) \text{ when } 0 < r < \beta\}\,.$$

It is easy to see that F is closed. It suffices to prove that $\mu(F) = 0$ since the set where $\underline{D}_h(\mu, [x, \infty[, x) > 0$ may be written as a countable union of this kind of sets. Assume contrary to the theorem that $\mu(F) > 0$. By Lemma 1.2, there is $x \in F$ such that $\underline{D}_h(\mu, F, x) = \underline{D}_h(\mu, x) = c < \infty$. If $0 < \varepsilon < \alpha/(2c)$, we may choose $0 < r < \min\{r_0, \beta\}$ so that

$$(1 - \varepsilon)c\, h(r) < \mu(F \cap [x - r, x + r]) \leq \mu([x - r, x + r]) < (1 + \varepsilon)c\, h(r).$$

Then also

$$\mu([x - r, x + r] \setminus F) < 2\, c\, \varepsilon\, h(r). \tag{2.1}$$

We will next show that $\mathcal{L}(F \cap [x, x + r]) = 0$. If $\gamma > 0$, then by the assumption (h2) there is $0 < \eta < \beta$ such that $r < \gamma\, h(r)$ for all $0 < r < \eta$. Now we may choose disjoint intervals $\{[x_i - r_i, x_i + r_i]\}_{i=1}^{\infty}$ so that $\mathcal{L}(F \cap [x, x + r] \setminus \bigcup_i [x_i - r_i, x_i + r_i]) = 0$, $x_i \in F \cap [x, x + r]$ and $r_i < \eta$ for all i. Thus

$$\begin{aligned}\mathcal{L}(F \cap [x, x + r]) &\leq 2 \sum_{i=1}^{\infty} r_i \leq 2\,\gamma \sum_{i=1}^{\infty} h(r_i) \leq 2\,\gamma\,\alpha^{-1} \sum_{i=1}^{\infty} \mu([x_i, x_i + r_i]) \\ &\leq 2\,\gamma\,\alpha^{-1} \mu([x - \eta, x + r + \eta]) \longrightarrow 0,\end{aligned}$$

as $\gamma \to 0$.

Write

$$[x, x + r[\setminus F = \bigcup_{i=1}^{\infty}]y_i, y_i + \delta_i[,$$

where intervals $]y_i, y_i + \delta_i[$ are disjoint, $y_i \in F$ for all i, and $\sum_{i=1}^{\infty} \delta_i = r$. This may be done since $x \in F$ and $\mathcal{L}([x, x + r] \cap F) = 0$. Using (h3), we obtain (Notice that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$ by (h1))

$$\mu([x - r, x + r] \setminus F) \geq \mu([x, x + r[\setminus F) = \sum_{i=1}^{\infty} \mu(]y_i, y_i + \delta_i[)$$

$$\geq \alpha \sum_{i=1}^{\infty} h(\delta_i) \geq \alpha\, h(r) > 2\, c\, \varepsilon\, h(r)$$

contrary to (2.1).

$\square$

Remarks 2.2. *1.* The above theorem is meaningful in the case (1); if $\underline{D}_h(\mu, x) = 0$ almost everywhere, then the assertion is trivial.
2. The main applications of Theorem 2.1 are the generalised Hausdorff and packing measures, $\mathcal{H}_h$ and $\mathcal{P}_h$, which are constructed using a gauge function h satisfying (h1)–(h3). If $A, B \subset \mathbb{R}$, $0 < \mathcal{H}_h(A) < \infty$, $0 < \mathcal{P}_h(A) < \infty$, $\mu = \mathcal{H}_h \llcorner A$, and $\nu = \mathcal{P}_h \llcorner B$, then (h3) implies that $\overline{D}_h(\mu, x) < \infty$ for μ-almost all $x \in \mathbb{R}$ and $\underline{D}_h(\nu, x) < \infty$ for ν-almost all $x \in \mathbb{R}$, see [16]. Among others, the functions $h(r) = r^s$ $(0 < s < 1)$, $h(r) = r \log(1/r)$, $h(r) = r \log(\log(1/r))$ etc. satisfy (h1)–(h3). In general, (h2) and (h3) are true if h is differentiable, h' is decreasing, and $\lim_{r \downarrow 0} h'(r) = \infty$. If h is concave, then (h3) is always true.
3. A method similar to that in [3] may be used to prove the statement of Theorem 2.1 if for all $\alpha > 0$ corresponds $\varepsilon = \varepsilon(\alpha) > 0$, $0 < r_1 = r_1(\alpha) \leq r_0$, and $0 < t = t(\alpha) < 1$ so that $\alpha h(tr) > h(r) - h(r - tr) + \varepsilon h(r)$ whenever $0 < r < r_1$. The idea is as follows: Suppose that h fulfils the above condition, and μ satisfies $\underline{D}_h(\mu, x) < \infty$ almost everywhere. Assume contrary that $\underline{D}_h(\mu,]x, \infty[, x) > d > 0$ in a set of positive μ measure. Then, using Lemma 1.2, we may find arbitrarily small intervals $[x - r, x + r] \subset \mathbb{R}$ and points $y \in [x, x + r]$ with $|x + r - y| \geq tr$ so that $\mu([x - r, y[) > (1 - \varepsilon)\underline{D}_h(\mu, x)h(r - tr)$ and $\mu([y, x + r]) > dh(tr)$. This leads to $\mu([x - r, x + r]) > (1 - \varepsilon)\underline{D}_h(\mu, x)h(r - tr) + d\,h(tr)$ which contradicts our assumptions for a suitably chosen α and r small. However, many interesting gauge functions such as $h(r) = r \log(1/r)$ fail to satisfy the condition desired above and for these functions this method is useless. The proof given in [1] for (1.1) works under the assumptions (h1)–(h3) if $\mu = \mathcal{H}_h \llcorner A$ and $0 < \mathcal{H}_h(A) < \infty$.
4. It is shown in [16, Example 2.12] that there are functions satisfying conditions (h1) and (h2) for which the claim of Theorem 2.1 fails. It is unknown if (h3) could be replaced by a doubling condition.

3. Upper densities

In this section we prove a very general one-sided upper density theorem for measures on $\mathbb{R}$. It shows that all measures on the real line are, in a sense, symmetric. This result was mentioned in [16] but it's proof has not appeared anywhere. Our proof is straightforward and it shows that on the real line upper densities are often easier to handle than lower densities; in the following theorem we have no assumptions for the function h. However, we assume that our measure is locally finite. If μ is not locally finite we have to put additional assumptions on h and μ. For example, if μ is non-atomic and satisfies $\underline{D}_h(\mu, x) < \infty$ for μ-almost all $x \in \mathbb{R}$, then the statement is still true. Our proof is influenced by those in [10] where related questions have been studied.

Theorem 3.1. *If μ is a locally finite measure on $\mathbb{R}$, then for any $h\colon]0, r_0[\to]0, \infty[$,*

$$\overline{D}_h(\mu, [x, \infty[, x) = \overline{D}_h(\mu,]-\infty, x], x) \geq \frac{1}{2}\,\overline{D}_h(\mu, x).$$

Proof. As in the previous proof, we may assume that μ is Borel regular. One can use standard arguments to show that the functions $x \mapsto \overline{D}_h(\mu, x)$, $x \mapsto \overline{D}_h(\mu, [x, \infty[, x)$, and $x \mapsto \overline{D}_h(\mu,]-\infty, x], x)$ are Borel functions. Let

$$A = \{x \in \mathbb{R} : \overline{D}_h(\mu, x) < \infty\}. \tag{3.1}$$

We shall first show that $\overline{D}_h(\mu, [x, \infty[, x) \geq \overline{D}_h(\mu, x)/2$ almost everywhere on the set A. If this is not the case, then we can find $0 < c < \infty$ and a Borel set $B \subset A$ with $\mu(B) > 0$ such that $\overline{D}_h(\mu, [x, \infty[, x) < c$ and $\overline{D}_h(\mu, x) > 2c$ for all $x \in B$. Moreover, choosing $0 < r_1 < r_0$ small enough and using the Borel regularity of μ (see [12, Theorem 1.10]), we find a closed set $C \subset B$ with $\mu(C) > 0$ such that

$$\mu([x, x + r]) < ch(r) \text{ for all } x \in C \text{ and } 0 < r < r_1. \tag{3.2}$$

By Lemma 1.2 we may find $x \in C$ such that $\overline{D}_h(\mu, C, x) > 2c$. Take $0 < r < r_1$ so that $\mu(C \cap [x - r, x + r]) > 2ch(r)$ and choose $y \in \{x, x - r\}$ for which $\mu(C \cap [y, y + r]) > ch(r)$. If we let $z = \min C \cap [y, y + r]$, then $\mu([z, z + r]) \geq \mu(C \cap [y, y + r]) > ch(r)$ contrary to (3.2). Thus $\overline{D}_h(\mu, [x, \infty[, x) \geq \overline{D}_h(\mu, x)/2$ for μ-almost every $x \in A$.

Next we shall show that $\overline{D}_h(\mu, [x, \infty[, x) = \infty$ almost everywhere on the set

$$D = \mathbb{R} \setminus A = \{x \in \mathbb{R} : \overline{D}_h(\mu, x) = \infty\}. \tag{3.3}$$

If this is not true, we may use a similar argument as above to find $0 < M < \infty$, $0 < r_2 < r_0$, and a closed set $E \subset D$ so that $\mu([x, x + r]) < Mh(r)$ for all $x \in E$ and $0 < r < r_2$. By Lemma 1.2, we find $x \in E$ and $0 < r < r_2$ so that $\mu(E \cap [x - r, x + r]) > 2Mh(r)$. Choosing $y \in \{x, x - r\}$ with $\mu(E \cap [y, y + r]) > Mh(r)$ and putting $z = \min E \cap [y, y + r]$ gives $\mu([z, z + r]) \geq \mu(E \cap [y, y + r]) > Mh(r)$ leading to a contradiction. Thus $\overline{D}_h(\mu, [x, \infty[, x) = \infty$ for μ-almost every $x \in D$ and consequently $\overline{D}_h(\mu, [x, \infty[, x) \geq \overline{D}_h(\mu, x)/2$ for μ-almost all $x \in \mathbb{R} = A \cup D$.

It remains to prove that $\overline{D}_h(\mu, [x, \infty[, x) = \overline{D}_h(\mu,]-\infty, x], x)$ almost everywhere and by symmetry it suffices to show that

$$\overline{D}_h(\mu, [x, \infty[, x) \leq \overline{D}_h(\mu,]-\infty, x], x) \tag{3.4}$$

holds for μ-almost every $x \in \mathbb{R}$. As it has already been proved that $\overline{D}_h(\mu, [x, \infty[, x) = \infty$, by symmetry it may be also shown that $\overline{D}_h(\mu,]-\infty, x], x) = \infty$ for μ-almost every $x \in D$ and thus it is enough to prove that (3.4) holds for μ-almost all $x \in A$. If this fails, arguing as before, we find $0 < r_3 < r_0$, $c > 0$, and a closed set $F \subset A$ with $\mu(F) > 0$ so that $\overline{D}_h(\mu, [x, \infty[, x) > c$ for all $x \in F$ and

$$\mu([x - r, x]) < ch(r) \text{ for all } x \in F \text{ and } 0 < r < r_3. \tag{3.5}$$

Since $\overline{D}_h(\mu,x)<\infty$ for $x\in F$, we see that

$$\begin{aligned}\overline{D}_h(\mu,\mathbb{R}\setminus F,x)&=\limsup_{r\downarrow 0}\frac{\mu((\mathbb{R}\setminus F)\cap[x-r,x+r])}{h(r)}\\&=\limsup_{r\downarrow 0}\frac{\mu([x-r,x+r])}{h(r)}\frac{\mu((\mathbb{R}\setminus F)\cap[x-r,x+r])}{\mu([x-r,x+r])}\\&=\overline{D}_h(\mu,x)\lim_{r\downarrow 0}\frac{\mu((\mathbb{R}\setminus F)\cap[x-r,x+r])}{\mu([x-r,x+r])}=0\end{aligned}$$

for μ-almost all $x\in F$ (see [12, Corollary 2.14]). This yields

$$\begin{aligned}\overline{D}_h(\mu,F\cap[x,\infty[,x)&=\limsup_{r\downarrow 0}\frac{\mu(F\cap[x,x+r])}{h(r)}\\&=\limsup_{r\downarrow 0}\left(\frac{\mu([x,x+r])}{h(r)}-\frac{\mu((\mathbb{R}\setminus F)\cap[x,x+r])}{h(r)}\right)\\&=\overline{D}_h(\mu,[x,\infty[,x)\end{aligned}$$

for μ-almost every $x\in F$. Thus we may find $x\in F$ and $0<r<r_3$ so that $\mu(F\cap[x,x+r])>ch(r)$. If $z=\max(F\cap[x,x+r])$, then $\mu([z-r,z])\geq\mu(F\cap[x,x+r])>ch(r)$ contrary to (3.5). □

Remarks 3.2. *1.* A slight modification in the argument shows that the constant $1/2$ in Theorem 3.1 may be replaced by $\liminf_{r\downarrow 0}h(r)/h(2r)$. Of course, this is an improvement only in the case when $\liminf_{r\downarrow 0}h(r)/h(2r)>1/2$.
2. If $h(r)=r^s$ $(0<s<1)$, then combining the above remark and the density bound $\overline{D}_h(\mathcal{H}^s,A,x)\geq 1$ for $\mathcal{H}^s$-almost all $x\in A$ provided $0<\mathcal{H}^s(A)<\infty$ yields $\overline{D}_h(\mathcal{H}^s\llcorner A,[x,\infty[,x)=\overline{D}_h(\mathcal{H}^s\llcorner A,]-\infty,x],x)\geq 2^{-s}$ which is still weaker than (1.2). However, if $h:[0,r_0[\to[0,\infty[$ is a function with $h(0)=0$, $A\subset\mathbb{R}$ with $0<\mathcal{H}_h(A)<\infty$, then the proof given in [1] in the case $h(r)=r^s$ may be generalised to show that $\overline{D}_h(\mathcal{H}_h\llcorner A,[x,\infty[,x)=\overline{D}_h(\mathcal{H}_h\llcorner A,]-\infty,x],x)=1$ for $\mathcal{H}_h$-almost all $x\in A$.
3. The above proof relies strongly on the geometry of the real line and rather different methods are needed when proving upper conical density theorems for measures in higher dimensions.
4. Theorem 3.1 is meaningful in all cases (1), (2), and (3). In particular, it says that $\overline{D}_h(\mu,[x,\infty[,x)=\overline{D}_h(\mu,]-\infty,x],x)=\infty$ almost everywhere if $\overline{D}_h(\mu,x)=\infty$ almost everywhere. This is the case often for the measures $\mathcal{P}^s\llcorner A$ where $\mathcal{P}^s$ is the s-dimensional packing measure, $A\subset\mathbb{R}$ with $0<\mathcal{P}^s(A)<\infty$, and $h(r)=r^s$.

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Department of Mathematics and Statistics P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland,

E-mail address: visuomal@maths.jyu.fi