Random measures, intersections, and applications

Ville Suomala
joint work with Pablo Shmerkin

University of Oulu, Finland

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Motivation

• Probabilistic constructions of sets and measures in Euclidean spaces arise in many problems of geometry and analysis. They give rise to ‘random’ objects which are often more regular than ones obtained via ‘deterministic’ constructions.

• We are mainly interested in intersection properties of random (fractal) sets and measures with deterministic sets.

• According to classical results, if $A$ is one of the following random subsets of $\mathbb{R}^d$
  1. a Brownian path,
  2. a random similar image of a fixed set $A_0$,
  3. fractal percolation limit set,
  and, if $B$ is a deterministic set, then $A \cap B$ is ‘typically’ nonempty (and has dimension $\dim A + \dim B - d$) if $\dim A + \dim B > d$. 
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Goals

Our goal is to find random sets/measures such that the above claim holds simultaneously for large parametrised families \( \{B_t\} \) of sets \( B_t \) and apply the result for various problems in analysis and geometry:

1. Dimension of non-tube null sets.
2. Uniform Marstrand-Mattila type projection results, without 'exceptional directions'.
3. Expected intersection size with all algebraic curves, all self-similar sets, etc.
4. Absolute continuity of convolutions of
   - random measures and deterministic measures.
   - two independent random measures.
   - self-convolutions.
   - Applications to sumsets...
5. Fourier decay for random measures. New examples of Salem sets/measures.
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A class of martingale measures

We call a random sequence \((\mu_n : \mathbb{R}^d \to [0, +\infty))\) of densities a **spatially independent (SI) martingale**, if

1. \(\mu_0\) is a deterministic bounded density with bounded support.

2. There exists an increasing sequence of \(\sigma\)-algebras \(\mathcal{B}_n\) such that \(\mu_n\) is \(\mathcal{B}_n\)-measurable (i.e. \(\mu_n(B)\) is \(\mathcal{B}_n\)-measurable for all Borel sets \(B\)). Moreover, for all \(x \in \mathbb{R}^d\),

\[
\mathbb{E}(\mu_{n+1}(x)|\mathcal{B}_n) = \mu_n(x).
\]

3. There is \(C > 0\) such that almost surely \(\mu_{n+1}(x) \leq C \mu_n(x)\) for all \(n\) and all \(x\).

4. There is \(C < \infty\) such that for any \((C 2^{-n})\)-separated family \(Q\) of sets of diameter \(2^{-n}\), the restrictions \(\{\mu_{n+1}|_Q|\mathcal{B}_n\}\) are independent.
• If \((\mu_n)\) is an (SI)-martingale, the sequence \(\mu_n\) almost surely converges to a limit measure \(\mu_\infty\).

• We are interested in the geometric properties of both \((\mu_n)\) and \(\mu_\infty\).
Examples of SI-martingales: Random cut-outs

We say that a sequence \((\mu_n)\) of measures on \(\mathbb{R}^d\) is of \((\mathcal{F}, \alpha, \eta)\)-cutout type, where \(\mathcal{F}\) is a collection of compact sets and \(\alpha, \eta > 0\), if there are numbers \(\delta, C > 0\) and a set \(\Omega \in \mathcal{F}\), such that

\[
\mu_n = 2^{\alpha n} 1[\Omega \setminus \bigcup_{j=1}^{M_n} \Lambda_j^{(n)}],
\]

for some random subset \(\{\Lambda_j^{(n)}\}_{j=1}^{M_n}\) of \(\mathcal{F}\), and, moreover, there is a finite random variable \(N_0\) such that \(M_n \leq C 2^{\eta n}\) for all \(n \geq N_0\), and \(N_0\) satisfies the sub-polynomial tail estimate

\[
P(N_0 > N) \leq C \exp(-N^\delta).
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In many cases, e.g. if \((\Lambda_n)_n\) are chosen according to a Poisson point process with a scale and translation invariant intensity measure on \(\mathbb{R}^d\), these cut-out measures are (SI)-martingales.
Ball type cut-outs
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Cell-type random measures

Let $\mathcal{F} \subset \mathcal{X}$ and $\tau, \eta > 0$. We say that a sequence $(\mu_n)$ of random measures is of $(\mathcal{F}, \tau, \eta)$-cell type, if there are $C, \delta > 0$ such that

$$\mu_n = \sum_{j=1}^{M_n} c_j 1[F_j], \quad (2)$$

where:

1. $\left\{ F_j^{(n)} \right\}_{j=1}^{M_n}$ is a random subset of $\mathcal{F}$,

2. The random variables $c_j = c_j^{(n)}$ satisfy $0 \leq c_j \leq C 2^{\tau n}$ for all $j, n$ almost surely,

3. Almost surely there is $N_0$ such that $M_n \leq C 2^{\eta n}$ for all $n \geq N_0$. Moreover, $\mathbb{P}(N_0 > N) \leq C \exp(-N^\delta)$. 

(SI)-martingales of this type include many natural subdivision random fractals like fractal percolation.
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Intersections with deterministic measures

Let \( \{ \eta_t \} \), \( t \in \Gamma \), be a family of measures indexed by a metric space \( (\Gamma, d) \) and let \( \{ \mu_n \}_n \) be an (SI)-martingale. For all \( t \in \Gamma \), and \( n \in \mathbb{N} \), we define a measure \( \mu_t^t \) as

- \( \mu_t^t(A) = \int_A \mu_n(x) \, d\eta_t(x) \),
- \( |\mu_t^t| = \mu_t^t(\mathbb{R}^d) \),

and further

- \( |\mu^t| = \lim_{n \to \infty} |\mu_n^t| \),

if the limit exists.
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(Sometimes, we also consider the measures \( \mu^t \) defined as weak limits of \( \mu^t_n. \))
Some examples of $\{\eta_t\}$

Let $\Omega \subset \mathbb{R}^d$ be fixed compact set (the support of $\mu_0$)

- For some $1 \leq k < d$, $\Gamma$ is the subset of **affine $k$-planes** which intersect $\Omega$, with the induced natural metric, and $\eta_V$ is $k$-dimensional Hausdorff measure restricted to $V \cap \Omega$.

- Given some $k \in \mathbb{N}$, $\Gamma$ is the family of all **algebraic curves** in $\mathbb{R}^2$ of degree at most $k$ which intersect $\Omega$, $d$ is the natural metric, and $\eta_\gamma$ is length measure on $\gamma \cap \Omega$.

- Let $m \geq 2$, and let $\Gamma$ be a totally bounded subset of uniformly contractive **self similar IFSs** with $m$ maps. Suppose that each IFS $(g_1, \ldots, g_m) \in \Gamma$ satisfies the OSC. The measure $\eta(g_1, \ldots, g_m)$ is the natural self-similar measure for the corresponding IFS.
Theorem

Let \( \{\mu_n\}_{n \in \mathbb{N}} \) be an (SI)-martingale, and let \( \{\eta_t\}_{t \in \Gamma} \) be a family of measures indexed by \((\Gamma, d)\). We assume that there are positive constants \( \alpha, s, \theta, \gamma_0 \) such that \( s > \alpha \) and so that the following holds:

1. \( \Gamma \) has finite box-dimension.
2. \( \eta_t(B(x, r)) = O(r^s) \) for all \( t \in \Gamma, x \in \mathbb{R}^d \), and \( 0 < r < 1 \).
3. Almost surely, \( \mu_n(x) \leq 2^{\alpha n} \) for all \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^d \).
4. Almost surely, there is a (random) integer \( N \), such that

\[
\sup_{t, u \in \Gamma, t \neq u; n \geq N} \frac{||\mu_n^t| - |\mu_n^u||}{2^{\theta n} d(t, u)^{\gamma_0}} < \infty.
\]  

(3)

Then there is \( \gamma > 0 \) (depending on all parameters) such that almost surely \( |\mu_n^t| \) converges uniformly in \( t \), exponentially fast, and the function \( t \rightarrow |\mu^t| \) is Hölder continuous with exponent \( \gamma \).
• Suppose we want to apply the previous theorem for a (SI)-martingale of \((\mathcal{F}, \alpha, \eta)\)-cutout or \((\mathcal{F}, \tau, \eta)\)-cell type.

• The assumption (1) is a technical condition that can be verified in 'most cases'.

• Likewise, it is usually a priori clear what is the optimal Frostman exponent in (2).

• (3) is saying that \(\dim \mu + \dim \eta_t > d\).

• The last condition (4) looks a bit technical, but it turns out that it is strongly related to the geometry of the shapes \(\Lambda \in \mathcal{F}\), more precisely on the modulus of continuity of

\[
t \mapsto \eta_t(\Lambda),
\]

for the sets \(\Lambda \in \mathcal{F}\).
\[ V_t \cap \{ d(x, \Lambda) \geq \varepsilon \} \]
Hölder continuity of orthogonal projections

**Theorem**

Let $\mu_n$ be a (SI)-martingale which is of $(\mathcal{F}, \alpha, \eta)$-cutout type, or of $(\mathcal{F}, \tau, \eta)$-cell type and satisfies $\mu_n(x) \leq 2^{\alpha n}$ for all $n, x$. Let $k \in \{1, \ldots, d-1\}$ with $\alpha < k$. Suppose that for some constants $0 < \gamma_0, C < \infty$ for all $V \in \Lambda_{d,k}$, all $\varepsilon > 0$ and any isometry $f$ which is $\varepsilon$ close to the identity, we have

$$\mathcal{H}^k (V \cap \Lambda \setminus f(\Lambda)) \leq C \varepsilon^{\gamma_0} \text{ for all } \Lambda \in \mathcal{F}.$$  

Then there is $\gamma > 0$ such that

(i) The sequence $|\mu^V_n| := \int_V \mu_n \, d\mathcal{H}^k$ converges uniformly over all $V \in \Lambda_{d,k}$. Denote the limit by $|\mu^V|$.

(ii) $||\mu^V| - |\mu^W|| \leq K d(V, W)^\gamma$.

(iii) For each $W \in \mathcal{G}_{d,d-k}$, the projection $P_W \mu_\infty$ is absolutely continuous. Moreover, if we denote its density by $f_W$, then the map $(x, W) \mapsto f_W(x)$ is Hölder continuous with exponent $\gamma$. 

Corollary

Let $A = \text{spt } \mu_\infty$. Under the assumptions of the previous theorem, and almost surely on $\mu_\infty \neq 0$, $P_V(A)$ has nonempty interior for all $V \in G_{d,d-k}$. 
Projection theorem without exceptional directions

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Let $A = \text{spt} \mu_\infty$. Under the assumptions of the previous theorem, and almost surely on $\mu_\infty \neq 0$, $P_V(A)$ has nonempty interior for all $V \in G_{d,d-k}$.

- The above corollary concerns the case $\dim_H \mu > d - k$. If $\alpha \geq k$, we show that almost surely, $\dim(P_V \mu_\infty, x) \geq d - \alpha$ for all $V \in G_{d,d-k}$ and $P_V \mu_\infty$-almost all $x \in V$. 
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- The above corollary concerns the case \( \dim H \mu > d - k \). If \( \alpha \geq k \), we show that almost surely, \( \dim(P_V\mu_\infty, x) \geq d - \alpha \) for all \( V \in G_{d,d-k} \) and \( P_V\mu_\infty \)-almost all \( x \in V \).
- In particular, almost surely on \( \mu_\infty \neq 0 \), we have \( \dim_H(P_V(A)) \geq d - \alpha \) for all \( V \).
- Under slightly stronger assumptions we can show that a.s \( \dim_B(A \cap V) \leq k - \alpha \) for all \( V \in G_{d,d-k} \).

and with positive probability, \( \dim_H(A \cap V) = k - \alpha \) for an open set of \( V \):s.
• For fractal percolation in $\mathbb{R}^2$, the above results are known to hold (Falconer-Grimmett 1992, Rams-Simon 2013, 2014, Peres-Rams).

• Note however, that for fractal percolation, the horizontal and vertical directions are exceptional for the absolute (and Hölder) continuity of the projections.
Intersections with more general families of sets/measures

Theorem

If \( d = 2 \) and \((\mu_n)\) is a both an (SI)-martingale, and of cut-out type such that each \( \Lambda \in \mathcal{F} \) is a similar image of the Von Koch snowflake, then all the statements of the previous theorem hold if \( G_{d,d-k} \) is replaced by any family \( \mathcal{V} \) of algebraic curves of bounded degree.
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In particular, if $\dim \mu > 1$, then a.s. each polynomial image $P(A)$, $P : \mathbb{R}^2 \to \mathbb{R}$, contains an interval.
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The following holds almost surely, if $\mu_n$ is both an (SI)-martingale and of cut-out type, such that each $\Lambda \in \mathcal{F}$ is a ball: If the almost sure dimension of $\mu$ (and $A = \text{spt} \mu$) is $s$, then a.s. for all self-similar sets $E \subset \mathbb{R}^d$ satisfying the open set condition, we have

$$\dim_B(E \cap A) \leq \max\{0, \dim E + s - d\}.$$
Products, convolutions, etc.

Under various (but rather general), geometric conditions of the shapes of $\Lambda \in \mathcal{F}$, we show that for our random measures $\mu_\infty$ (with a.s. dimension $s \in (0, \infty)$):

- For any deterministic measure $\nu$ on $\mathbb{R}$ with $\nu(B(x, r)) = O(r^t)$ for some $t > 1 - s$, the convolutions $\mu_\infty * (r \nu)$ are absolutely continuous for all $r \neq 0$ and if $f_r$ denotes the density of $\mu_\infty * (r \nu)$, the map $(x, r) \mapsto f_r(x)$ is locally Hölder continuous.

- In particular, if $\dim H E > 1 - s$, then a.s. all the sets $A + rE$, $r \neq 0$ contain an interval.

- If $(\mu'\infty)$ and $(\mu''\infty)$ are independent realisations of the same random measure in $\mathbb{R}^d$ with $s > d/2$, the same holds a.s. for all $\mu'\infty * S (\mu''\infty)$, $S \in \text{GL}(\mathbb{R}^d)$.

- If $s > d/2$, the same holds for all $\mu_\infty * S (\mu_\infty)$ apart from $S = -rI_d$, $r > 0$. 
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- If \( s > d/2 \), the same holds for all \( \mu_\infty * S(\mu_\infty) \) apart from \( S = -rI_d, r > 0 \).
• For dyadic subdivision fractals $\mu_n$ on $[0, 1]$ (including fractal percolation), we show that they are a.s Salem measures:

$$\hat{\mu}(\xi) = O \left( (1 + |\xi|)^{-\sigma/2} \right),$$

for any $\sigma < \dim_H \mu$. 
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for any \(\sigma < \dim_H \mu\).

• Combining this with the result that if \(\alpha < 1/2\), then \(\mu_\infty * \mu_\infty\) is a.s. locally Hölder (and thus bounded), it follows that these measures satisfy a Fourier restriction estimate

\[
\|\hat{f}d\mu\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^2(\mu)}
\]

for all \(p \geq 4\). This partially answers a question of I. Laba.
Further applications

• Existence of non tube-null sets of given dimension $s \geq d - 1$ (Shmerkin and S. 2012)

• Orthogonal projections of random covering sets (Chen, Koivusalo, Li, S. 2013)

• Convolutions of $m \geq 3$ random measures $\mu_1^\infty, \ldots, \mu_m^\infty$.

• Dimension threshold for the existence of given patterns (all progressions, all angles etc) inside random fractal sets.

• The last two applications require a more general version of the main continuity result, allowing certain degree of dependenced along lower dimensional subplanes. (work in progress)
Thank you!