Spatially independent martingales: Intersections, and Applications

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Motivation

- Probabilistic constructions of sets and measures in Euclidean spaces arise in many problems of geometry and analysis. They give rise to ‘random’ objects which are often more regular than ones obtained via ‘deterministic’ constructions.
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- We are mainly interested in intersection properties of random (fractal) sets and measures with deterministic sets/measures.

- According to classical results, if $A$ is one of the following random subsets of $\mathbb{R}^d$
  1. a Brownian path,
  2. a random similar image of a fixed set $A_0$,
  3. fractal percolation limit set,
and, if $B$ is a deterministic set, then $A \cap B$ is ‘typically’ nonempty (and has dimension $\dim A + \dim B - d$) if $\dim A + \dim B > d$. 
Goals

Our goal is to find random sets/measures such that the above claim holds simultaneously for large parametrised families \( \{B_t\} \) of sets \( B_t \) and apply the result for various problems in analysis and geometry:

1. Dimension of non-tube null sets.
2. Uniform Marstrand-Mattila type projection results, without 'exceptional directions'.
3. 'Expected' intersection size with all algebraic curves, all self-similar sets, etc.
4. Arithmetic structure in random sets.
5. Absolute continuity of convolutions of random measures.
6. Fourier decay for random measures.
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A class of martingale measures

We call a random sequence \((\mu_n : \mathbb{R}^d \to [0, +\infty))\) of densities a spatially independent (SI) martingale, if

1. \(\mu_0\) is a deterministic bounded density with bounded support.

2. There exists an increasing sequence of \(\sigma\)-algebras \(B_n\) such that \(\mu_n\) is \(B_n\)-measurable (i.e. \(\mu_n(B)\) is \(B_n\)-measurable for all Borel sets \(B\)). Moreover, for all \(x \in \mathbb{R}^d\),

\[
\mathbb{E}(\mu_{n+1}(x) | B_n) = \mu_n(x).
\]

3. There is \(C > 0\) such that almost surely \(\mu_{n+1}(x) \leq C\mu_n(x)\) for all \(n\) and all \(x\).

4. There is \(C < \infty\) such that for any \((C2^{-n})\)-separated family \(Q\) of sets of diameter \(2^{-n}\), the restrictions \(\{\mu_{n+1}|_Q | B_n\}\) are independent.
• If \((\mu_n)\) is an (SI)-martingale, the sequence \(\mu_n\) almost surely converges to a limit measure \(\mu_\infty\).

• We are interested in the geometric properties of both \((\mu_n)\) and \(\mu_\infty\).

• Examples of SI-martingales include e.g. random multiplicative cascades, Poissonian cut-out measures (random soups) and their various generalisations.
Outline of the method:

Suppose \((\mu_n)\) is an SI-martingale supported in the unit cube and let \(\eta_t, t \in \Gamma\) be a parametrised family of measures in \(\mathbb{R}^d\). Suppose that for some \(\alpha < s\),

\[
\mu_n(x) \leq C 2^{n\alpha} \text{ for all } x \in \mathbb{R}^d, n \in \mathbb{N} \\
\eta_t(B(x, r)) \leq C r^s \text{ for all } x \in \mathbb{R}^d, r > 0 \text{ and } t \in \Gamma.
\]

For each dyadic cube \(Q \in \mathcal{Q}_n\) denote by \(X^t_Q\) the random variable

\[
X^t_Q = \int_Q (\mu_{n+1}(x) - \mu_n(x)) \, d\eta_t(x).
\]

Then for a fixed \(t\), \(X_Q\) are zero mean independent r.v. and

\[
\left\|\mu_{n+1} \cap \eta_t\right\| - \left\|\mu_n \cap \eta_t\right\| = \sum_{Q \in \mathcal{Q}_n} X_Q.
\]
A large deviation estimate allows to conclude that a.s. for each fixed $t$ the total mass

$$
\|\mu_n \cap \eta_t\| := \int \mu_n(x) \, d\eta_t(x)
$$

converges (extremely fast) to a limit $Y^t$.

In many situations, the maps $t \mapsto \|\mu_n \cap \eta_t\|$ are continuous wrt. to $t$ with a reasonable modulus of continuity (deterministic or random).

Combining these two observations, it is possible to conclude a.s. continuity of $t \mapsto Y^t$ for the limits $Y^t = \lim_{n \to \infty} \|\mu_n \cap \eta_t\|$.

We define $\mu_\infty \cap \eta_t = \lim_n \mu_n \cap \eta_t$ so that $Y_t = \|\mu_\infty \cap \eta_t\|$. 
Dyadic subdivision fractals

Denote $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$ be the family of dyadic subcubes of $[0, 1]^d$. Let $\{W_Q\}_{Q \in \mathcal{D}}$ be random variables such that:

1. The law of $W_Q, Q \in \mathcal{D}_{n+1}$ is measurable with respect to the $\sigma$-algebra $\mathcal{B}_n$ generated by $\{W_Q\}_{Q \in \mathcal{Q}_k, k \leq n}$.
2. Almost surely $W_Q \in [0, C]$ for all $Q \in \mathcal{Q}$.
3. $\mathbb{E}(W_Q | \mathcal{B}_n) = 1$ for all $Q \in \mathcal{Q}$.
4. If $\{Q_j\} \subset \mathcal{Q}_{n+1}$ and the $Q_j$ are subsets of pairwise disjoint cubes in $\mathcal{Q}_n$, then $W_{Q_j}$ are independent.

We define a sequence $(\mu_n)$ as follows. Let $\mu_0 = 1_{[0,1]^d}$. For $n \geq 0$, set

$$
\mu_{n+1}(x) = W_Q \mu_n(x),
$$

where $Q \in \mathcal{Q}_{n+1}$ is the cube containing $x$.

Then $(\mu_n)$ is an SI-martingale.
An example

Fractal percolation: There is a fixed $0 < p < 1$ such that for each $Q \in \mathcal{Q}$:

$$W_Q = p^{-1} \text{ with probability } p,$$
$$W_Q = 0 \text{ with probability } 1 - p.$$
Theorem A

Let $\mu_n$ be an SI-martingale satisfying certain geometric assumptions (in particular, it could be one of the dyadic subdivision fractals defined above). Suppose

$$\mu_n(x) \leq C 2^{\alpha n} \quad \text{for some} \quad \alpha < \frac{d}{2}.$$ 

Then almost surely, the convolution

$$\mu_\infty \ast S \mu_\infty$$

is absolutely continuous for each $S \in \text{GL}_d(\mathbb{R})$. Moreover, if the density is denoted by $f_S$, then the map $(S, x) \mapsto f_S(x)$ is jointly locally Hölder continuous (with a quantitative exponent) on

$$(\text{GL}_d(\mathbb{R}) \setminus \{S \in \text{GL}_d(\mathbb{R}) : S + I_d \text{ is not invertible.}\}) \times \mathbb{R}^d.$$
Key idea in the proof:

The convolution $\mu_\infty \ast S\mu_\infty$ is the image of the product $\mu_\infty \times \mu_\infty$ under the projection $g_S(x, y) = x + Sy$.

It can be shown that (if $S + I_d$ is invertible) the density of $\mu_\infty \ast S\mu_\infty$ is given by the total mass

$$Y^{S,z} := \| (\mu_\infty \times \mu_\infty) \cap \eta_{S,z} \|$$

where $\eta_{S,z}$ is the uniform distribution ($\mathcal{H}^d$-measure) on the fibre $g_S^{-1}(z)$.

Since these fibres intersect the dyadic cubes of $\mathbb{R}^d \times \mathbb{R}^d$ (or more generally, the cells of the underlying SI-martingale) transversally, we are able to conclude a.s. the desired Hölder continuity for $(S, z) \mapsto Y^{S,z}$. 
Fourier Decay of DSM

For a measure $\mu$ supported in $[0, 1]^d$, its Fourier dimension is

$$\dim_F \mu = \sup\{0 \leq \sigma \leq d : \hat{\mu}(\xi) \leq C_\sigma (1 + |\xi|)^{-\sigma/2}\}$$

where $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} \exp(-2\pi i y \cdot \xi) \, d\mu(y)$.

We say that $\mu$ is a Salem measure if $\dim_F \mu = \dim_H \mu$ (it is well known that $\dim_F \mu \leq \dim_H \mu$).

There are relatively few known examples of Salem measures (most of them random).
Theorem B

Let $\mu_n$ be a dyadic subdivision martingale, and suppose for each $n$ there is $p_n \in (0, 1)$ such that $W_F \in \{0, p_n^{-1}\}$ for all $F \in \mathcal{F}_n$. Write $\beta_n = (p_1 \cdots p_n)^{-1}$, and note that in this case

$$\mu_n = \beta_n 1_{A_n}$$

for some random set $A_n \subset [0, 1]^d$.

Suppose $\frac{1}{n} \log_2 \beta_n \to \alpha \in (d - 2, d]$. Then a.s. $\mu_\infty$ is a Salem measure and

$$\dim_F \mu_\infty = \dim_H \mu_\infty = \dim_H \text{supp } \mu_\infty = \dim_B \text{supp } \mu_\infty = d - \alpha.$$
New examples of Salem measures

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In particular, this implies that Fractal percolation limit sets are a.s. Salem (if their dimension is $\leq 2$).
Key idea in the proof:

Consider $\hat{\mu}_n(\xi)$ as an intersection of $\mu_n$ and the (complex) measure

$$d\eta_\xi x = \exp(-2\pi ix \cdot \xi) \, dx.$$
Restriction estimates for fractal measures

Recall that the restriction problem for a measure $\mu$ on Euclidean space asks for what pairs $p, q$ there is an estimate

$$\| f d\mu \|_{L^p(\mathbb{R}^d)} \leq C_{p,q} \| f \|_{L^q(\mu)} \cdot$$  \hspace{1cm} (1)

**Theorem (Mitsis, Mockenhaupt, (Bak and Seeger))**

Suppose that for all $x, \xi \in \mathbb{R}^d$, $r > 0$:

$$\mu(\mathcal{B}(x,r)) \leq C r^{\alpha},$$ \hspace{1cm} (2)

$$|\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\beta/2}.$$ \hspace{1cm} (3)

Then (1) with $q = 2$ holds whenever $p \geq 2 \left( \frac{2}{d} - 2 \alpha + \beta \right)$.

If $\mu$ satisfies (2), (3) for $\alpha$ and $\beta$ arbitrarily close to the Hausdorff dimension of $\text{supp} \mu$, we say that $\mu$ is a strong Salem measure.
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$$\|\hat{f}d\mu\|_{L^p(\mathbb{R}^d)} \leq C_{p,q}\|f\|_{L^q(\mu)}.$$  \hspace{1cm} (1)

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Suppose that for all $x, \xi \in \mathbb{R}^d, r > 0$:

$$\mu(B(x, r)) \leq Cr^\alpha,$$  \hspace{1cm} (2)

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Then (1) with $q = 2$ holds whenever $p \geq \frac{2(2d-2\alpha+\beta)}{\beta}$. 

Restriction estimates for fractal measures

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In $d = 1$ the range of the Mockenhaupt’s theorem is sharp:

**Theorem (Hambrook and Laba 2013, Chen 2014)**

For any $1 \leq p < 2 + \frac{4(1-\alpha)}{\beta}$, there are measures satisfying (2), (3) such that (1) fails.
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I. Laba proposed the following open question:
Is it possible that there are some fractal measures in $\mathbb{R}$ satisfying (2), (3) such that (1) holds with some exponent $p < \frac{2(2d-2\alpha+\beta)}{\beta}$?
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**Theorem (X. Chen 2014)**

If $\mu$ has the property that $\mu \ast \mu$ is absolutely continuous with a bounded density, then (1) holds for $p \geq 4$ and $q \geq \frac{p}{(p-2)}$, (and in particular for $q = 2$.)
Using the above result of Chen as a black box and combining Theorems A and B, we obtain a positive answer to the question of Laba:

**Theorem**

Let $\alpha_0 \in (1/2, 1)$. Then there is a strong Salem measure $\mu$ in $\mathbb{R}$ supported on a set of Hausdorff dimension $\alpha_0$, such that $\mu * \mu$ has a Hölder continuous density, and (1) holds for all $p \geq 4$ and $q \geq p/(p - 2)$.

Note that for $d = 1$ the range of Mockenhaupt’s Theorem in the strong Salem case is $p \in (4/\alpha_0 - 2, \infty)$, and $4/\alpha_0 - 2 > 4$ for $\alpha_0 < 2/3$. 
Thank you!