LOCAL DISTRIBUTION OF FRACTAL SETS AND MEASURES

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Academic dissertation

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Jyväskylä, December 2004

Ville Suomala

List of included articles

This thesis consists of an introductory part and the following three publications:

[A] **On the conical density properties of measures on $\mathbb{R}^n$.**
Ville Suomala.
To appear in *Math. Proc. Cambridge Philos. Soc.*

[B] **Nonsymmetric conical upper density and k-porosity.**
Antti Käenmäki and Ville Suomala.
Preprint 299, Department of Mathematics and Statistics, University of Jyväskylä, 2004.

[C] **Tangential behavior of functions and conical densities of Hausdorff measures.**
Ville Suomala.
Preprint 303, Department of Mathematics and Statistics, University of Jyväskylä, 2004.

In the introductory part these articles will be referred as [A], [B], and [C]. The article [B] is an outcome of a collaboration to which both authors have contributed equally.

1. Introduction

The birth of geometric measure theory dates back to 1920's and 1930's when Besicovitch started the systematic study of sets A having positive and finite s-dimensional Hausdorff measure, $0 < \mathcal{H}^s(A) < \infty$. The main emphasis was given to plane sets having positive and finite 1-dimensional measure, and to sets on the real line having positive and finite s-dimensional measure for some $0 < s < 1$. In particular, Besicovitch studied the distribution of the above sets A on certain cones. These results have since then been used and also generalised by various authors, but mostly only for the Hausdorff measures. In this thesis we consider more general measures and study their distribution on cones.

Among the most useful concepts in geometric measure theory, or fractal geometry in general, are Hausdorff measures and the Hausdorff dimension. However, during the last decades there has been a growing interest in fractals, mostly due to their applicability in modelling scientific phenomena, and this has led mathematicians to develop also new kinds of tools for studying them. For example, instead of Hausdorff dimension it is sometimes more advantageous to consider some other dimension, such as Minkowski dimension or packing dimension. The role of using different measures has also become very important. When studying a particular fractal set, it is often useful to study measures supported on this set rather than the set itself since the measures carry information about the geometry of the set. It is thus natural and important to figure out the structure of these measures.

One of the most important sample results for the Hausdorff measures is the following: If A is a subset of the Euclidean n-space $\mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$ for some $0 < s < n$, then

$$1 \leq \limsup_{r \downarrow 0} \mathcal{H}^s(B(x,r) \cap A)/r^s \leq 2^s \tag{1.1}$$

for $\mathcal{H}^s$-almost all points $x \in A$ where $B(x,r)$ denotes the open ball with centre at x and radius r. Given this result we know roughly how much of mass there is in some small balls. It is then natural to ask how the set A (or the measure $\mathcal{H}^s \llcorner A$, given by $\mathcal{H}^s \llcorner A(B) = \mathcal{H}^s(A \cap B)$ for $B \subset \mathbb{R}^n$) is distributed around the points x. Among others, Besicovitch [2], [3], Marstrand [17], Federer [8], Salli [28], and Mattila [22], have studied this kind of questions by considering the upper and lower limits of the ratio $\mathcal{H}^s(B(x,r) \cap A \cap C)/r^s$, where C is a cone around x, see (2.1). The main motivation of the research carried out in [A] was to find out if these, so called conical density theorems, can be obtained if the Hausdorff measures are replaced by more general measures. A special interest was focused on the conical density properties of packing measures, $\mathcal{P}^s$, which are often thought as dual objects to Hausdorff measures. The results of [A] may also be applied to a large collection of general Hausdorff- and packing type measures. In particular, one can use also gauge functions other than power functions.

The local distribution of a given fractal can be approached from two different viewpoints. On one hand, we can study the distribution of this fractal when the amount of mass is known in small balls. The problems studied in [A] fall into this category. On the other hand, it is sometimes possible to go to the opposite direction as well, that is, to deduce information on the dimension (on the amount of mass on small balls), from information that is given on the distribution of the mass. In this context porosity has proved to be a useful tool. Originating from the works of Denjoy [4] and Dolzenko [5], porosity has traditionally been used in analysis to describe the smallness of certain exceptional sets, see [31]. Porosity of a set describes the maximal relative size of holes on this set on small scales, and intuitively it is clear that the bigger the porosity is, the smaller the dimension should be. Various theorems concerning this statement have been proved for example by Sarvas [30], Martio and Vuorinen [20], Mattila [22], Salli [29], and Koskela and Rohde [15]. A porosity of a measure was introduced by Eckmann, Järvenpää, and Järvenpää in [6] and its effect on dimension has been studied also by Järvenpää and Järvenpää [12] and by Beliaev and Smirnov [1]. If a set has holes in several different directions instead of just one, this does not affect the value of the porosity. Consequently, the best theoretical upper bound for the dimension of a porous set can never be better than $n-1$ where n is the dimension of the ambient space; any $(n-1)$-dimensional affine subspace has maximal porosity. This fact motivated the definition of k-porosity introduced in [B]. It is proved in [B] that if a set $A \subset \mathbb{R}^n$ has big holes into k orthogonal directions at all small scales, then the Hausdorff dimension of A, $\dim_{\mathrm{H}} A$, can not be much greater than $n-k$. The proof of this fact relies on a certain nonsymmetric conical upper density result for the Hausdorff measures which is also proved in [B].

In [C] a very special conical density problem is studied. By constructing a C^1-function with suitable properties, it is shown that for all integers $0 < m < n$, there are rectifiable m-dimensional sets $A \subset \mathbb{R}^n$ for which the m-lower density of A is strictly positive on all open half-spaces at $\mathcal{H}^m$-almost all points $x \in A$. This example led us also to study the differentiability structure of typical continuous functions. It is shown in [C] that for a typical continuous function $f : [0,1] \to \mathbb{R}$, all extended real numbers are symmetrical essential derived numbers of f at almost all points of the interval $(0,1)$.

2. Conical densities

A set $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$ will be called an s-set, recall the definition of $\mathcal{H}^s$ from 2.7. Note that we do not require an s-set to be measurable. For the Hausdorff measures, conical density theorems illustrate the distribution of s-sets on small scales. The lower density results allow one to find a lot of empty space around a typical point of an s-set. This kind of results were used for example by Marstrand [17], [18], [19]. He proved among other things that for any Radon measure μ on $\mathbb{R}^n$, the positive and finite density $\lim_{r\downarrow 0} \mu(B(x,r))/r^s$ can exist in a set of positive μ-measure only if s is an integer. In fact, a great deal more can be said about the distribution of such measures, see Preiss [27]. Upper density results for Hausdorff measures describe how much of an s-set is contained on certain narrow cones. They have been used in an essential way, for example, in connection with rectifiable and purely unrectifiable sets. The importance of the conical density theorems is based on the fact that they can be used to obtain geometric information for the measure from a given metric information, that is, the values of the measure on small balls reflect the distribution of the measure.

In this section we shall first give an overview of the conical density properties of Hausdorff measures and then discuss the new results proved in [A], [B], and [C]. For any integers $0 \leq m \leq n$, let $G(n,m)$ stand for the collection of all m-dimensional linear subspaces of $\mathbb{R}^n$ and for any $\theta \in S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$, denote $L_\theta = \{t\theta : t > 0\}$. When $x \in \mathbb{R}^n$, $\theta \in S^{n-1}$, $V \in G(n,m)$, and $0 \leq \eta \leq 1$, we define the following cones:

$$\begin{aligned} X(x,V,\eta) &= \{y \in \mathbb{R}^n : d(y-x,V) < \eta|y-x|\}, \\ X^+(x,\theta,\eta) &= \{y \in \mathbb{R}^n : d(y-x,L_\theta) < \eta|y-x|\}, \\ H(x,\theta,\eta) &= \{y \in \mathbb{R}^n : (y-x)\cdot\theta > \eta|y-x|\}, \\ H(x,\theta) &= H(x,\theta,0) = \{y \in \mathbb{R}^n : (y-x)\cdot\theta > 0\}. \end{aligned} \tag{2.1}$$

For $0 < r < \infty$, we also use the abbreviations $X(x,r,V,\eta) = X(x,V,\eta) \cap B(x,r)$, $X^+(x,r,\theta,\eta) = X^+(x,\theta,\eta) \cap B(x,r)$, $H(x,r,\theta,\eta) = H(x,\theta,\eta) \cap B(x,r)$, and $H(x,r,\theta) = H(x,\theta) \cap B(x,r)$. Note that $X^+(x,\theta,(1-\eta^2)^{1/2}) = H(x,\theta,\eta)$. However, both notations are useful for us.

Suppose that $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$. Let us focus on the lower and upper limits of the ratio $\mathcal{H}^s\left(B(x,r) \cap A \cap C(x)\right)/r^s$ where C is one of the cones in (2.1). On the real line the only cones are half-lines and in this case conical densities are one-sided densities. Besicovitch studied these densities on the real line and proved the following [2, Theorem 2].

Theorem 2.1. *If $0 < s < 1$ and $A \subset \mathbb{R}$ is an s-set, then*

$$\liminf_{r\downarrow 0} \mathcal{H}^s(A \cap (x,x+r))/r^s = \liminf_{r\downarrow 0} \mathcal{H}^s(A \cap (x-r,x))/r^s = 0$$

and

$$\limsup_{r\downarrow 0} \mathcal{H}^s(A \cap (x,x+r))/r^s = \limsup_{r\downarrow 0} \mathcal{H}^s(A \cap (x-r,x))/r^s = 1$$

for $\mathcal{H}^s$-almost all points $x \in A$.

On the real line, the upper one-sided densities are often easier to handle than the corresponding lower densities. See [21] and [A, Theorem 3.3] for upper density results of measures on the real line.

In the plane, the direct analogue of Theorem 2.1 is not true anymore. If we let $A = \{(t,0) \in \mathbb{R}^2 : 0 < t < 1\}$, $\theta = (1,0) \in S^1$, $V = \{(0,t) : t \in \mathbb{R}\} \in G(2,1)$, and $0 < \eta < 1$, then $\lim_{r\downarrow 0} \mathcal{H}^1(A \cap H(x,r,\theta,\eta))/r = 1$ and $\lim_{r\downarrow 0} \mathcal{H}^1(A \cap X(x,r,V,\eta))/r = 0$ for all $x \in A$. In fact, every rectifiable curve shares a somewhat similar behaviour. If the set A is purely unrectifiable, then the situation is different. We recall that a set $A \subset \mathbb{R}^n$ is called m-rectifiable if $\mathcal{H}^m$-almost all of it can be covered by countably many Lipschitz images of $\mathbb{R}^m$. A set $A \subset \mathbb{R}^n$ is purely m-unrectifiable if it intersects every Lipschitz graph of $\mathbb{R}^m$ in a set of zero $\mathcal{H}^m$ measure. The next theorem is also due to Besicovitch [3, Theorem 8, Theorem 13].

Theorem 2.2. *Let $A \subset \mathbb{R}^2$ be a purely 1-unrectifiable 1-set. If $\theta \in S^1$, $V \in G(2,1)$, and $0 < \eta < 1$, then*

$$\liminf_{r\downarrow 0} \mathcal{H}^1(A \cap H(x,r,\theta))/r = 0 \tag{2.2}$$

and

$$\limsup_{r\downarrow 0} \mathcal{H}^1(A \cap X(x,r,V,\eta))/r > c(\eta) \tag{2.3}$$

for $\mathcal{H}^1$-almost all points $x \in A$. Here $c(\eta) > 0$ is a constant depending only on η.

For the cones $H(x,r,\theta,\eta)$, $\eta > 0$, claim (2.2) was proved already by Gillis [10]. Besicovitch [3, p. 327–328] gave an example to illustrate that (2.3) is not true for any $0 < \eta < 1$ if the cones $X(x,r,V,\eta)$ are replaced by the one-sided cones $X^+(x,r,\theta,\eta)$. However, it holds for the cones $X^+(x,r,\theta,1) = H(x,r,\theta)$ [3, Theorem 7].

The study of the conical density properties of s-sets for nonintegral s was pioneered by Marstrand. We collect some of his results from [17] in the following theorem. See also Falconer [7, §4] and Mattila [23, §11].

Theorem 2.3. *Let $0 < s < 2$, $0 < \eta < 1$, and $A \subset \mathbb{R}^2$ with $0 < \mathcal{H}^s(A) < \infty$.*

1. *For $\mathcal{H}^s$-almost all points $x \in A$, there is a direction $\theta \in S^1$ such that $\liminf_{r\downarrow 0} \mathcal{H}^s(A \cap H(x,r,\theta,\eta))/r^s = 0$.*
2. *If $s < 1$ and $\theta \in S^1$, then $\liminf_{r\downarrow 0} \mathcal{H}^s(A \cap H(x,r,\theta))/r^s = 0$ for $\mathcal{H}^s$-almost all $x \in A$.*
3. *If $s > 1$, then for $\mathcal{H}^s$-almost all $x \in A$, there is a direction $\theta \in S^1$ such that $\liminf_{r\downarrow 0} \mathcal{H}^s(A \cap H(x,r,\theta))/r^s = 0$.*
4. *If $s > 1$ and $\theta \in S^1$, then $\limsup_{r\downarrow 0} \mathcal{H}^s(A \cap X^+(x,r,\theta,\eta))/r^s > c(\eta)$ for $\mathcal{H}^s$-almost all $x \in A$ where $c(\eta) > 0$ is a constant depending only on η.*

Marstrand's arguments can be easily generalised to prove that the direct analogues of statements 1 and 2 hold also in $\mathbb{R}^n$. Moreover, claim 3 holds for s-sets $A \subset \mathbb{R}^n$ if $n-1 < s < n$. Lorent [16] proved that one can choose the directions θ in 1 to lie on a given $(n-1)$-dimensional linear subspace, provided that either $s < n-1$, or $s = n-1$ and A is purely $(n-1)$-unrectifiable. The following generalisation of the statement 4 was given by Salli [28].

Theorem 2.4. *Suppose that $V \in G(n, n-m)$, $s > m$, $0 < \eta < 1$, and $A \subset \mathbb{R}^n$ is an s-set. Then*

$$\limsup_{r \downarrow 0} \mathcal{H}^s(A \cap X(x, r, V, \eta))/r^s > c_1 \tag{2.4}$$

for $\mathcal{H}^s$-almost all $x \in A$. If $n-1 < s < n$ and $\theta \in S^{n-1}$, then

$$\limsup_{r \downarrow 0} \mathcal{H}^s(A \cap X^+(x, r, \theta, \eta))/r^s > c_2 \tag{2.5}$$

for $\mathcal{H}^s$-almost all $x \in A$. Above the constant $c_1 > 0$ depends only on n, m, s, and η, and $c_2 > 0$ depends only on n, s, and η.

In fact, Salli proved also much more general results. In (2.4) he was able to use cones generated by open sets $U \subset G(n, n-m)$, and in (2.5) he used cones generated by open sets of S^{n-1}, see [28, Theorem 3.7, Theorem 3.8]. Federer (see [9, §3.3.17]) has shown that claim (2.4) holds also for purely m-unrectifiable sets $A \subset \mathbb{R}^n$ if $0 < \mathcal{H}^m(A) < \infty$. See also Morse and Randolph [26]. As noted above, claim (2.5) is not true for purely $(n-1)$-unrectifiable $(n-1)$-sets.

The following result of Mattila [22] shows that it is not necessary to fix V in Theorem 2.4. Below, $\gamma_{n,n-m}$ denotes the unique Radon probability measure on $G(n, n-m)$ which is invariant under the orthogonal group $O(n)$, see [23, §3.9]. Denote also $C_x = \{x\} + \bigcup_{V \in C} V$ if $C \subset G(n, n-m)$, and $C_x = \{x\} + \bigcup_{\theta \in C} L_\theta$ for $C \subset S^{n-1}$.

Theorem 2.5. *Let $s > m$, $0 < \eta < 1$, and $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$. Then for $\mathcal{H}^s$-almost all points $x \in A$,*

$$\limsup_{r \downarrow 0} \inf_C \mathcal{H}^s(A \cap B(x, r) \cap C_x)/r^s > c \tag{2.6}$$

where the infimum is taken over all Borel sets $C \subset G(n, n-m)$ for which $\gamma_{n,n-m}(C) > \eta$, and $c > 0$ is a constant depending only on n, m, s, and η. Moreover, if $n-1 < s < n$, then the infimum in (2.6) may be taken over the Borel sets $C \subset S^{n-1}$ for which $\mathcal{H}^{n-1}(C) > \eta$.

The purpose of the paper [A] was to find out how much of the above mentioned conical density results remain true if Hausdorff measures are replaced by more general measures. It was proved useful to study measures μ for which there is a function $h\colon (0, r_0) \to (0, \infty)$ that may be used to estimate the measures of some small balls around typical points $x \in \mathbb{R}^n$. One of the main results proved in [A] is the following generalisation for the claim 1 of Theorem 2.3.

Theorem 2.6 ([A] Theorem 2.1)**.** *Let $m, n \in \mathbb{N}$ with $m \leq n$. Assume that $h\colon (0, r_0) \to (0, \infty)$ fulfils the following three conditions:*

$$\lim_{r \downarrow 0} h(r) = 0, \tag{h1}$$

$$\lim_{r \downarrow 0} h(r)/r^m = \infty, \tag{h2}$$

$$h(r_1) + h(r_2) \geq h((r_1^m + r_2^m)^{1/m}) \text{ whenever } r_1^m + r_2^m \leq r_0^m. \tag{h3}$$

Suppose that $V \in G(n, m)$, $\eta > 0$, and μ is a measure on $\mathbb{R}^n$ which satisfies $\liminf_{r \downarrow 0} \mu(B(x, r))/h(r) < \infty$ for μ-almost all $x \in \mathbb{R}^n$. For μ-almost all $x \in \mathbb{R}^n$, there is $\theta = \theta(x) \in V \cap S^{n-1}$ so that $\liminf_{r \downarrow 0} \mu(H(x, r, \theta, \eta))/h(r) = 0$.

At first, the above theorem might seem a little technical, so let us illustrate it in a light of some examples. Let $0 < s < n$, and take an s-set $A \subset \mathbb{R}^n$. If $V = \mathbb{R}^n$, $\mu = \mathcal{H}^s \llcorner A$, and $h(r) = r^s$, then the assumptions of Theorem 2.6 are clearly satisfied. Thus, Theorem 2.6 really implies the claim 1 of Theorem 2.3. If $\mathcal{P}^s$ denotes the s-dimensional packing measure, see the definition below, and $\mu = \mathcal{P}^s \llcorner A$ where $0 < \mathcal{P}^s(A) < \infty$, then $\liminf_{r \downarrow 0} \mu(B(x, r))/r^s = 2^s$ for $\mathcal{P}^s$-almost all $x \in A$, see [23, Theorem 6.10]. Hence, Theorem 2.6 shows that the statement 1 of Theorem 2.3 is true also for the packing measures $\mathcal{P}^s$. Suppose now that $s < m < n$, and let again $\mu = \mathcal{H}^s \llcorner A$, where $A \subset \mathbb{R}^n$ is an s-set. Now our theorem asserts that for $\mathcal{H}^s$-almost all $x \in A$, the lower density $\liminf_{r \downarrow 0} \mu(H(x, r, \theta, \eta))/r^s$ is zero for some $\theta \in V \cap S^{n-1}$ whenever $V \in G(n, m)$. Thus we can find many "empty" sectors around x instead of just one. It is worth mentioning that this is not true anymore if $s \geq m$, see [A, Example 2.5].

We recall the definitions of the following generalised Hausdorff and packing type measures.

Definition 2.7. Suppose that $h\colon [0, r_0] \to [0, \infty)$ is a function with $h(0) = 0$. For any $A \subset \mathbb{R}^n$ and $0 < \delta \leq \infty$, we define

$$\mathcal{H}_h^\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} h\left(\operatorname{diam}(A_i)\right) \,:\, A \subset \bigcup_{i=1}^{\infty} A_i \text{ and } \operatorname{diam}(A_i) < \delta \text{ for all } i \right\}$$

and

$$P_h^\delta(A) = \sup \sum_i h\left(\operatorname{diam}(B_i)\right)$$

where the supremum is taken over all disjoint collections of balls $B_1, B_2, \ldots$ whose centres lie on A and for which $0 \leq \operatorname{diam}(B_i) < \delta$ for all i. Define $\mathcal{H}_h(A) = \lim_{\delta \downarrow 0} \mathcal{H}_h^\delta(A)$ and $P_h(A) = \lim_{\delta \downarrow 0} P_h^\delta(A)$. Now $\mathcal{H}_h$ is called the h-Hausdorff measure and defining

$$\mathcal{P}_h(A) = \inf \left\{ \sum_{i=1}^{\infty} P_h(A_i) \,:\, A \subset \bigcup_{i=1}^{\infty} A_i \right\}$$

for $A \subset \mathbb{R}^n$, gives the h-packing measure, $\mathcal{P}_h$. If $h(r) = r^s$, then we denote $\mathcal{H}_h = \mathcal{H}^s$ and $\mathcal{P}_h = \mathcal{P}^s$, and call these the s-dimensional Hausdorff measure and the s-dimensional packing measure, respectively.

Suppose that h fulfils (h1)–(h3). Let $A, B \subset \mathbb{R}^n$, $0 < \mathcal{H}_h(A) < \infty$, $0 < \mathcal{P}_h(B) < \infty$, $\mu = \mathcal{H}_h \llcorner A$, and $\nu = \mathcal{P}_h \llcorner B$. Then $\limsup_{r\downarrow 0} \mu(B(x,r))/h(r) \le \limsup_{r\downarrow 0} h(2r)/h(r)$ for μ-almost all $x \in \mathbb{R}^n$ and $\liminf_{r\downarrow 0} \nu(B(x,r))/h(r) \le \limsup_{r\downarrow 0} h(2r)/h(r)$ for ν-almost all $x \in \mathbb{R}^n$. These density estimates are proved like Theorems 6.2 and 6.10 in [23]. Since the condition (h3) implies that h is doubling, namely $\limsup_{r\downarrow 0} h(2r)/h(r) \le 2^m < \infty$, our theorem applies for the measures μ and ν.

We just saw that if h satisfies the conditions (h1)–(h3), then there are many natural measures to which Theorem 2.6 can be applied. Let us now briefly discuss these assumptions. The condition (h2) says that $h(r)$ tends to zero quite slowly compared to r^m. This is needed for example to exclude the measures $\mathcal{H}^m \llcorner V$, where $V \in G(n,m)$. Assumption (h1) is used for technical reasons. It guarantees that μ has no atoms. If (h1) is not true, then the assertion of Theorem 2.6 is trivially true. In [A, Example 2.12] it is shown that that there are functions h satisfying (h1) and (h2) for which the claim of Theorem 2.6 fails and thus we need more assumptions. However, the condition (h3) is somewhat technical, and it is reasonable to ask if it could be replaced for example by a doubling condition on h. Fortunately, the condition (h3) is fulfilled by many natural gauge functions. Among others, the functions $h(r) = r^s \log(1/r)$, $h(r) = r^s \log(\log(1/r))$ etc. satisfy (h1)–(h3) when $0 < s \le m$. The case $s = n$ is of particular interest: Let $h(r) = r^n \log(1/r)$. Then h is only slightly bigger than r^n, in particular all the sets with $0 < \mathcal{H}_h(A) < \infty$ are of Hausdorff dimension n. Still, according to Theorem 2.6, one can find sectors $H(x, r_i, \theta, \eta)$ around typical points $x \in A$ such that their measure is small compared to $h(r_i)$. In this case one can even take $\eta = 0$, see [A, p. 5].

If μ and h satisfy the assumptions of Theorem 2.6 with $m = 1$, then one can fix θ and let $\eta = 0$. The next result generalises the statement 2 of Theorem 2.3.

Theorem 2.8 ([A] Theorem 2.2)**.** *Suppose that $h\colon (0, r_0) \to (0, \infty)$ fulfils the conditions* (h1)–(h3) *with $m = 1$ and let μ be a measure on $\mathbb{R}^n$ such that $\liminf_{r\downarrow 0} \mu(B(x,r))/h(r) < \infty$ for μ-almost all $x \in \mathbb{R}^n$. If $\theta \in S^{n-1}$, then $\liminf_{r\downarrow 0} \mu(H(x,r,\theta))/h(r) = 0$ for μ-almost all $x \in \mathbb{R}^n$.*

Using the same technique as in the proof of Theorem 2.6, one can also obtain the following result. It has been proved by Lorent [16] in the case $m = n-1$. The latter statement is (2.2) in $\mathbb{R}^n$, and it can be proved by generalising Besicovitch's argument from [3].

Theorem 2.9 ([A] Theorem 2.11)**.** *Suppose that $0 \le m < n$, $A \subset \mathbb{R}^n$ is purely m-unrectifiable with $0 < \mathcal{H}^m(A) < \infty$, and $V \in G(n,m)$. If $0 < \eta < 1$, then for*

$\mathcal{H}^m$-almost all $x \in A$, there is $\theta \in V \cap S^{n-1}$ for which

$$\liminf_{r \downarrow 0} \mathcal{H}^m(A \cap H(x,r,\theta,\eta))/r^m = 0.$$

Moreover, if $m = 1$ and $\theta \in S^{n-1}$ is fixed, then

$$\liminf_{r \downarrow 0} \mathcal{H}^1(A \cap H(x,r,\theta))/r = 0$$

for $\mathcal{H}^1$-almost all $x \in A$.

Due to a simple compactness argument, the directions $\theta = \theta(x) \in V \cap S^{n-1}$ in Theorems 2.3 (part 1), 2.6, and 2.9 may be chosen to be independent of the opening angle, that is, the same direction works for all $0 < \eta < 1$. One of the interesting open questions concerning the lower conical densities is that when can one take $\eta = 0$ in Theorem 2.6. Even for the measures $\mathcal{H}^s$, there is no any satisfactory answer known. It is only known that for any integers $m \leq n \in \mathbb{N}$ there is an m-rectifiable m-set $A \subset \mathbb{R}^n$ so that for $\mathcal{H}^m$-almost all $x \in A$, $\liminf_{r \downarrow 0} \mathcal{H}^m(A \cap H(x,r,\theta))/r^m > 0$ for all $\theta \in S^{n-1}$. If $A \subset \mathbb{R}^n$ is an s-set and $s \in (1,n)$ is either nonintegral, or if s is an integer and A is purely s-unrectifiable, then it is not known whether for $\mathcal{H}^s$-almost all $x \in A$, there is $\theta = \theta(x) \in S^{n-1}$ with $\liminf_{r \downarrow 0} \mathcal{H}^s(A \cap H(x,r,\theta))/r^s = 0$. These questions have been discussed in [C, §3], see also §4 below.

Another possible research direction in the future would be to study which measures μ on $\mathbb{R}^n$ have the following property: For μ-almost all $x \in \mathbb{R}^n$, there is $\theta \in S^{n-1}$ so that $\liminf_{r \downarrow 0} \mu(H(x,r,\theta,\eta))/\mu(B(x,r)) = 0$ for all $0 < \eta < 1$. If $h\colon (0,r_0) \to \mathbb{R}$ fulfils (h1)–(h3) and if $0 < \liminf_{r \downarrow 0} \mu(B(x,r))/h(r) < \infty$ for μ-almost all $x \in \mathbb{R}^n$, then the above property holds by Theorem 2.6. On the other hand, it seems to be unknown if the statement is true for the measures $\mathcal{H}^s \llcorner A$ when $A \subset \mathbb{R}^n$ is an s-set with $\liminf_{r \downarrow 0} \mathcal{H}^s(A \cap B(x,r))/r^s = 0$ for $\mathcal{H}^s$-almost all $x \in A$. If μ is doubling, the above condition is true if and only if a certain upper porosity of μ equals 1. This follows from the arguments used by Mera and Morán in [25].

In [A, §3] the methods of Salli from [28] were developed to prove some generalisations of Theorem 2.4. The first one is an analogue of (2.4).

Theorem 2.10 ([A] Theorem 3.1)**.** *Let $h\colon (0,r_0) \to (0,\infty)$ be a function which satisfies, for some $0 < c_1 < \infty$, the doubling condition*

$$h(2r) < c_1 h(r). \tag{d}$$

Assume also that for some $m \in \mathbb{N}$ and for all $r < r_0$, $0 < t < 1$,

$$h(tr) \leq t^m h(r), \tag{h4}$$

$$\lim_{r \downarrow 0} h(r)/r^m = 0. \tag{h5}$$

If μ is a measure on $\mathbb{R}^n$ such that $\liminf_{r\downarrow 0} \mu(B(x,r))/h(r) < \infty$ *for μ-almost all $x \in \mathbb{R}^n$, $V \in G(n, n-m)$, and $0 < \eta \leq 1$, then*

$$\limsup_{r\downarrow 0} \mu(X(x,r,V,\eta))/h(r) \geq c \limsup_{r\downarrow 0} \mu(B(x,r))/h(r) \tag{2.7}$$

for μ-almost all $x \in \mathbb{R}^n$. Above the constant $c > 0$ depends only on m, n, η, and c_1.

The main applications of Theorem 2.10 are the generalised Hausdorff- and packing measures, $\mathcal{H}_h$ and $\mathcal{P}_h$, which are constructed using a gauge function h that satisfies the assumptions (d), (h4), and (h5). Such functions include for example $h(r) = r^s$ $(m < s < n)$, $h(r) = r^s \log(1/r)$ $(m < s < n)$, and $h(r) = r^m/\log(1/r)$. The last one of these is interesting since it is only slightly smaller than r^m: If $h(r) = r^m/\log(1/r)$, then all the sets with $0 < \mathcal{H}_h(A) < \infty$ have Hausdorff dimension m. Slightly more restrictive assumptions were needed to show that when $m = n-1$ in the above theorem, then the cones $X(x,r,V,\eta)$ may be replaced by their one side. For example, the function $h(r) = r^s \log(1/r)$ satisfies the assumptions of the next theorem if $s > n-1$ while the function $h(r) = r^{n-1}/\log(1/r)$ does not.

Theorem 2.11 ([A] Theorem 3.2)**.** *Assume that $h\colon (0,r_0) \to (0,\infty)$ satisfies the doubling condition* (d)*. Suppose also that for some $s > n-1$ and for all $r < r_0$, $0 < t < 1$,*

$$h(tr) \leq t^s h(r). \tag{h6}$$

If μ is a measure on $\mathbb{R}^n$ with $\liminf_{r\downarrow 0} \mu(B(x,r))/h(r) < \infty$ *for μ-almost all $x \in \mathbb{R}^n$, $\theta \in S^{n-1}$, and $0 < \eta \leq 1$, then*

$$\limsup_{r\downarrow 0} \mu(X^+(x,r,\theta,\eta))/h(r) \geq c \limsup_{r\downarrow 0} \mu(B(x,r))/h(r) \tag{2.8}$$

for μ-almost all $x \in \mathbb{R}^n$. Above the constant $c > 0$ depends only on n, s, η, and c_1 (the constant of (d)*).*

In Theorems 2.10 and 2.11 it is possible that $\limsup_{r\downarrow 0} \mu(B(x,r))/h(r) = \infty$ almost everywhere. This is often the case for the packing measures $\mu = \mathcal{P}^s \llcorner A$ when $0 < \mathcal{P}^s(A) < \infty$. In such a case Theorems 2.10 and 2.11 say that the upper densities in the narrow cones of (2.7) and (2.8), respectively, are also infinite almost everywhere.

The study of k-porous sets, see §3 below, led us to study upper densities in more general nonsymmetric cones. The next theorem should be compared with Theorem 2.5. It is proved in [B] only for the Hausdorff measures but the same proof gives also the following more general result. Figure 1 illustrates the cones $X(x,r,V,\alpha) \setminus H(x,\theta,\eta)$.

Theorem 2.12 ([B] Theorem 2.6)**.** *Suppose that $0 \leq m < n$ and let $h\colon (0,r_0) \to (0,\infty)$ be a function with*

$$h(\epsilon r)/\epsilon^m h(r) \longrightarrow 0 \qquad \textit{uniformly for all } 0 < r < r_0 \tag{h7}$$

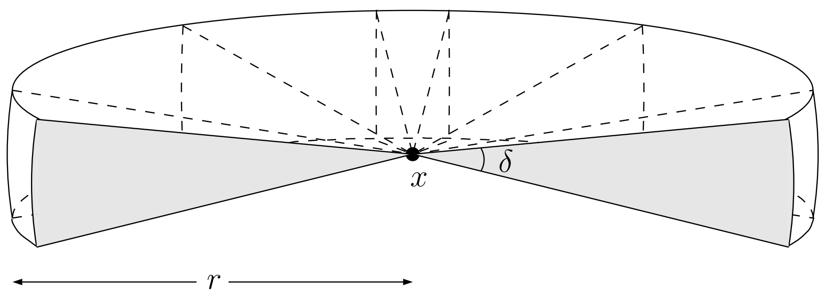

FIGURE 1. The set $X(x,r,V,\alpha) \setminus H(x,\theta,\eta)$ when $n = 3$, $m = 1$, and $\alpha = \sin(\delta/2)$.

as $\epsilon \downarrow 0$*. For all* $0 < \alpha, \eta \le 1$*, there is a constant* $c > 0$ *depending only on* n, m, h, α*, and* η *such that*

$$\limsup_{r\downarrow 0} \inf_{\substack{\theta\in S^{n-1}\\ V\in G(n,n-m)}} \frac{\mu\big(X(x,r,V,\alpha)\setminus H(x,\theta,\eta)\big)}{h(r)} \ge c \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{h(r)}$$

for μ *almost all* $x \in \mathbb{R}^n$ *whenever* μ *is a measure on* $\mathbb{R}^n$ *with*

$$\limsup_{r\downarrow 0} \mu(B(x,r))/h(r) < \infty \tag{2.9}$$

for μ *almost every* $x \in \mathbb{R}^n$*.*

Theorem 2.12 does not imply Theorem 2.5 since much more general cones than $X(x,r,V,\alpha)$ are used in Theorem 2.5. On the other hand, Theorem 2.5 does neither give Theorem 2.12 for the Hausdorff measures (unless $m = n-1$) since, in the case $m < n-1$, all the cones used in Theorem 2.5 are symmetric. The proof of Theorem 2.12 given in [B] is relatively simple compared to Mattila's proof of Theorem 2.5 which is a corollary to a sophisticated result dealing with the distribution of general Radon measures on $\mathbb{R}^n$. Theorem 2.12 is often more useful than Theorems 2.10 and 2.11 since we do not need to fix V in Theorem 2.12. However, Theorem 2.12 is not a generalisation of Theorems 2.10 and 2.11 due to the assumption (2.9). The condition (h7) is also stronger than the assumptions (h4) and (h5) in Theorem 2.10. It has been shown by Käenmäki and the author [14] that if h fulfils (h6) for some $s > m$, then the lim sup in (2.9) may be replaced by lim inf.

3. DIMENSION ESTIMATE FOR k-POROUS SETS

Porosity conditions are used to describe the relative size of the holes in a given set. One can find plenty of different definitions of porosity from the literature but if one wants to relate porosity to dimension, then it is worthwhile to use a definition of porosity which describes the size of the maximal holes at all, or at least most, small scales. Dimension estimates for porous sets have been used, for example, to study the boundary behaviour of quasiconformal mappings, see [20] and [15].

The following definition of k-porosity, por_k, was introduced in [B]. For $0 < k \leq n$, $x \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$, and $r > 0$, set

$$\begin{aligned}\mathrm{por}_k(A, x, r) = \sup\{\varrho :\ &\text{there are } z_1, \ldots, z_k \in \mathbb{R}^n \text{ such that } B(z_i, \varrho r) \subset \\ &B(x, r) \setminus A \text{ for every } i, \text{and } (z_i - x) \cdot (z_j - x) = 0 \text{ if } i \neq j\}.\end{aligned}$$

The *k-porosity* of A at a point x is $\mathrm{por}_k(A, x) = \liminf_{r \downarrow 0} \mathrm{por}_k(A, x, r)$, and the k-porosity of A is given by $\mathrm{por}_k(A) = \inf_{x \in A} \mathrm{por}_k(A, x)$. We also use the abbreviation $\mathrm{por} = \mathrm{por}_1$. The dimensional behaviour of 1-porous sets is rather well investigated. If the relative size of the holes is big, the following theorem may be used to estimate the dimension. Below $\dim_{\mathrm{p}}$ denotes the packing dimension, and $\dim_{\mathrm{M}}$ refers to the upper Minkowski dimension, see [23, §5].

Theorem 3.1. *For all $n \in \mathbb{N}$, there is a constant $c = c(n) < \infty$ so that*

$$\dim_{\mathrm{p}} A \leq n - 1 + c/\log(1/(1 - 2\alpha)) \tag{3.1}$$

whenever $0 < \alpha < 1/2$ and $A \subset \mathbb{R}^n$ with $\mathrm{por}(A) \geq \alpha$. Moreover, if A satisfies $\mathrm{por}(A, x, r) \geq \alpha$ for all $x \in A$ and $0 < r < r_0$, then the packing dimension in (3.1) may be replaced by the Minkowski dimension.

Theorem 3.1 is due to Salli [29]. It is asymptotically sharp modulo the constant c, see [29, §3.8.2]. An analogous result for the Hausdorff dimension was obtained earlier by Mattila [22, Corollary 3.4].

Every hyperplane has maximal 1-porosity and thus $n - 1$ is the best theoretical dimension bound that can be obtained from the information on por_1. The concept of k-porosity was introduced since it can be used to describe also sets whose dimension is smaller than $n - 1$. For instance, many familiar Cantor type constructions lead to k-porous sets. Let $0 < \lambda < 1/2$. If $C_\lambda \subset \mathbb{R}$ is the λ-Cantor set (see [23, §4.10]), then the set $A = C_\lambda \times C_\lambda \times \mathbb{R} \subset \mathbb{R}^3$ is 2-porous and the set $B = C_\lambda \times C_\lambda \times C_\lambda \subset \mathbb{R}^3$ is 3-porous. Moreover, $\mathrm{por}_2(A) \approx 1/2 - \lambda \approx \mathrm{por}_3(B)$. The following theorem gives an upper bound for the Hausdorff dimension of k-porous sets.

Theorem 3.2 ([B] Theorem 3.2)**.** *Suppose $k, n \in \mathbb{N}$ with $0 < k \leq n$. Then*

$$\sup\{s > 0 : \mathrm{por}_k(A) > \varrho \textit{ and } \dim_{\mathrm{H}}(A) > s \textit{ for some } A \subset \mathbb{R}^n\} \longrightarrow n - k$$

as $\varrho \to 1/2$.

In the case $k = 1$, Theorem 3.2 was obtained by Mattila [22] as a corollary to Theorem 2.5. We followed the same lines in [B] and proved Theorem 3.2 using Theorem 2.12. Quite recently, the above result with asymptotically sharp estimates was generalised for the packing and Minkowski dimensions by

E. Järvenpää, M. Järvenpää, A. Käenmäki, and the author [13]. A related approximation property and its effect on dimension has been studied by Mattila and Vuorinen [24].

In [B, §4] some dimension estimates for sets with porosity close to zero are discussed.

4. Symmetric essential derived numbers

The article [C] was motivated by the problem that when does the statement 1 of Theorem 2.3 hold with $\eta = 0$. Combining Theorems 2.2 (first claim) and statements 2–3 of Theorem 2.3, it follows that (in the plane) this may be done when $s \neq 1$ or if $s = 1$ and A is purely 1-unrectifiable.

For a differentiable function $f\colon (0,1) \to \mathbb{R}$, define the sets $A^+(f,x)$ and $A^-(f,x)$ by

$$A^+(f,x) = \{y \in (0,1) : f(y) > f(x) + f'(x)(y-x)\},$$
$$A^-(f,x) = \{y \in (0,1) : f(y) < f(x) + f'(x)(y-x)\}.$$

In [C, Theorem 2.1] we constructed a C^1-function $f : [0,1] \to \mathbb{R}$ so that for almost all $x \in (0,1)$, there is $r > 0$ for which $(x, x+r) \subset A^+(f,x)$ and $(x-r, x) \subset A^-(f,x)$. This shows, see [C,§3], that one can not always take $\eta = 0$ in Theorem 2.3 when $s = 1$ and A is purely 1-rectifiable.

The above example led us to study the distribution of the sets $A^+(f,x)$ and $A^-(f,x)$ for typical functions $f \in C^1[0,1]$. As usual, we say that some property is typical on the space $C[0,1]$ of continuous functions $f\colon [0,1] \to \mathbb{R}$ if it holds on a residual subset of $C[0,1]$ with respect to the norm $||f|| = \sup_{x\in[0,1]} |f(x)|$. On the space $C^1[0,1]$ typicality is defined similarly, the norm being defined by $||f|| = \sup_{x\in[0,1]} \max\{|f(x)|, |f'(x)|\}$. The following theorem shows that the example discussed above is exceptional in $C^1[0,1]$. Below $\mathcal{L}$ denotes the Lebesgue measure.

Theorem 4.1 ([C] Corollary 2.5). *For a typical $f \in C^1[0,1]$, it holds that*

$$\limsup_{r\downarrow 0} \mathcal{L}((x-r, x+r) \cap A^+(f,x))/(2r) = 1,$$
$$\limsup_{r\downarrow 0} \mathcal{L}((x-r, x+r) \cap A^-(f,x))/(2r) = 1,$$
$$\liminf_{r\downarrow 0} \mathcal{L}((x-r, x+r) \cap A^+(f,x))/(2r) = 0,$$
$$\liminf_{r\downarrow 0} \mathcal{L}((x-r, x+r) \cap A^-(f,x))/(2r) = 0,$$

for all $x \in (0,1) \setminus A$ where A is a set (depending on f) of Hausdorff dimension zero.

To prove Corollary 4.1 we had to study the differentiability structure of typical continuous functions. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ denote the set of the extended real numbers. We call a number $c \in \overline{\mathbb{R}}$ a symmetrical essential derived number of f at x provided there is a set $E \subset \mathbb{R}$ with

$$\limsup_{r \downarrow 0} \mathcal{L}((x-r, x+r) \cap E)/(2r) = 1 \tag{4.1}$$

for which

$$\lim_{\substack{y \to x \\ y \in E}} \frac{f(y) - f(x)}{y - x} = c. \tag{4.2}$$

Theorem 4.2 ([C] Theorem 2.3)**.** *For a typical $f \in C[0,1]$, every $c \in \overline{\mathbb{R}}$ is a symmetrical essential derived number of f at x for $\mathcal{L}$-almost every point on the interval* $(0,1)$.

I have not found a proof for Theorem 4.2 in the literature, although many related topics have been discussed. Jarnik [11] called $c \in \overline{\mathbb{R}}$ a right essential derived number of f at x if there is a set $E \subset \mathbb{R}$ for which (4.2) holds and $\limsup_{r \downarrow 0} \mathcal{L}((x, x+r) \cap E)/r = 1$. Left essential derived numbers are defined in an analogous way. A point $x \in \mathbb{R}$ is called a knot point of f if all extended real numbers are both left- and right essential derived numbers of f at x. Jarnik proved in [11] that almost all points $x \in (0,1)$ are knot points of f for a typical function $f \in C[0,1]$. Zajíček [32] generalised Jarník's result using porosity notations. In fact, it is possible to modify Zajíček's method to prove Theorem 4.2 and even more general results, see [C, Theorem 2.4]. These are needed to prove Corollary 4.1 and related more general results, see [C, Theorem 2.5].

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