ON THE DIMENSION OF SOLAR ATTRACTOR

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Abstract. Analyzing the average over a year (a) and over a month (b) of Wolf numbers and radiocarbon data (c), we have obtained the dimensions d of the solar attractor which are: 3.3 (a), 4.3 (b), 4.7 (c). During the Maunder minimum such a dimension turns out to be significantly higher: 8.0 (c); whereas during the period of a phase catastrophe (1792–1828) Wolf numbers averaged over a month yield $d = 3.0$ (b). We have also investigated the sensitivity of our inferences to the number of available experimental points. Positive values of the Kolmogorov entropy and first Lyapunov exponent explicitly show the stochastic behaviour of the Sun.

1. Introduction

Recent years have been marked by increased interest in the theory of nonlinear dynamical systems. Such deterministic systems manifest important coexisting features: both regular and stochastic. These properties may be clearly understood considering phase space trajectories which depict some phase surface, say a torus. If, for instance, a motion with a low frequency characterizes a regular component (a limit cycle), the trajectories of initially close points on the surface itself (the higher frequencies) may drastically diverge. If so, these directions are unstable and one cannot predict where our system is. This so-called deterministic chaos is to be discussed below (in what follows the terms 'chaotic' and 'stochastic' are equivalent). Notice the principle difference of stochastic properties considered and a random measurement noise. The latter can be extracted from a data series.

The above-mentioned fully refers to the Sun as a nonlinear generator of solar activity. In fact, the presence of regular periods in sunspot variations is well known, in particular the 11-year period. At the same time an amplitude and a phase of this cycle can not practically be predicted. Moreover, during the last ~ 8000 years there were several irregular minima of solar activity like those of Maunder, Spörer, and Wolf (Eddy, 1976). Such depressed activity resulted in solar modulation changes and hence in higher radiocarbon concentration in tree rings (Eddy, 1976; Kocharov *et al.*, 1983).

To construct a model of solar activity on a long time-scale one should take into account the above-mentioned factors. The most general features of a solar generator were discussed by Gudzenko and Chertoprud (1980), who found the mean 11-year cycle. In the papers of Ruzmaikin (1981) and Malinetzkii, Ruzmaikin, and Samarskii (1986), there was suggested an attractor which was a good theoretical fit to the minima of activity. A corresponding Lorenz-type system has a regime accounting for the deep minima of solar activity. In the present paper we evaluate a set of parameters that quantitatively indicate the character of this attractor. In our calculations we have used

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the time series $x(t)$ of monthly averaged Wolf numbers (1749–1987) and also of yearly averaged radiocarbon abundances in the Earth's atmosphere $\Delta^{14}C$ (1564–1900).

2. Methods and Results

The most important parameter which characterizes the attractor and its stochasticity is a dimension of the attractor provided it contains a fractional part. This value is close to the minimum number of independent variables needed to describe the system. The problem of choosing them is extremely complicated and we shall not be concerned with it in this paper.

First of all let us transform the time series $x(t)$ into the n-dimensional phase space vectors $\mathbf{X}_k \equiv \mathbf{X}(t_k) = \{x(t_k), \ldots, x(t_k + (n-1)\tau)\}$, where τ is a delay time. According to Grassberger and Procaccia (1983), the dimension d can be calculated as a slope of curves $\log_2 C(r) - \log_2 r$:

$$d(n) = \Delta \log_2 C(r)/\Delta \log_2 r\,, \tag{1}$$

where r is the hypersphere diameter; $C(r)$, correlation sum:

$$C(r) = \frac{1}{N^2} \sum_{i \neq j} \theta(r - \| \mathbf{X}_i - \mathbf{X}_j \|)\,, \tag{2}$$

θ being the Heaviside function, and N the total number of points. Supposing the attractor to be a geometric object, one can write $C(r) \sim r^{d(n)}$. Therefore, $d(n)$ should be calculated within a linear part of the $\log_2 C(r) - \log_2 r$ dependence for a chosen n. An increase in n leads to the growth of $d(n)$, and if $n \geq n_s$, $d(n)$ is saturated at $d \approx d_s$. The value of d_s is believed to be the correlation dimension of the attractor. Note that the embedding condition $n > 2d_s + 1$ has to be obeyed. In practice, however, this inequality is not very rigorous.

The typical graphs when considering this problem are shown in Figures 1 and 2. A slope of the rectilinear part of each curve is found by using a least-squares fit. This approach gives also a standard deviation in d which is roughly of the order of $\sim 10\%$. Unfortunately, the greater n the narrower a straight line segment is and, therefore, measuring d is also harder. Especially this comment refers to the high-dimensional attractors, because they require a very large embedding dimension n. The measurement noise level can be separated by considering the smallest r region. If for $r < r^*$, $C(r) \sim r^n$ then r^* could give a noise amplitude.

The basic series used is the Wolf numbers consisting of 2868 points. The slope of corresponding curves have two plateaus around $d \approx 4.3$ and $d \approx 1.6$ (Aimanova and Makarenko, 1988; Morfill and Voges, 1989). The former characterizes 'local' features of the attractor connected with its basic period (in our case it is the 11-year cycle) and the latter is a global modulation of it. Below we consider only the first one. One important question is what amount of experimental data within the basic period is enough for reliably determining d. Usually one uses thousands of points. As we know,

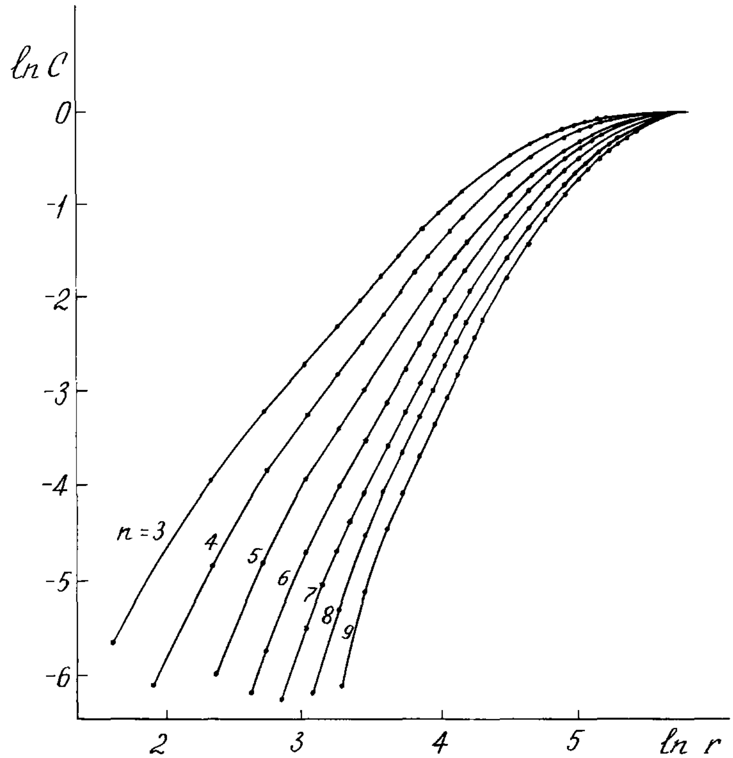

Fig. 1. The typical $\ln C - \ln r$ dependence for the Wolf numbers averaged over a year (1749–1987, data average (a), see Abstract).

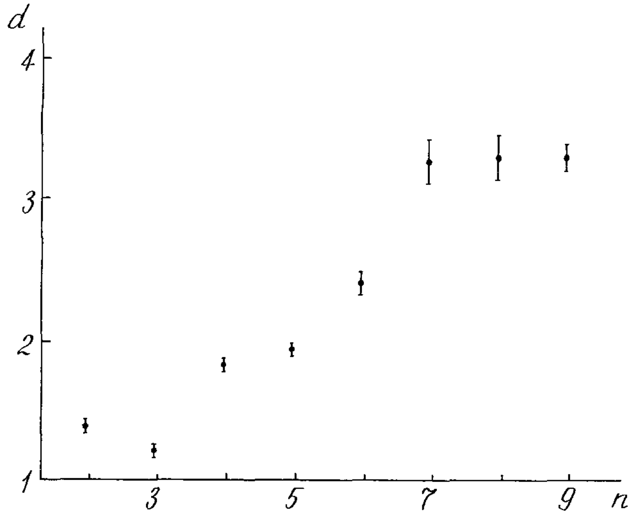

Fig. 2. The dimension of the attractor versus phase space embedding dimension n for the data of Figure 1. A saturation leads to the accurate d_s estimation, see text.

there has been no comprehensive investigation of that problem. Therefore, we tried to model artificially the lack of data. Our conclusion is that ~ 70 (or more) experimental points are sufficient to compute d with an accuracy better than 10%. This important inference was checked by means of attractors with known properties (Henon, Lorenz, and Rossler). In all cases a minimum number was approximately the same. We chose the smallest delay time τ needed to derive the maximum information from the data. One should remember that a phase space within its main period has to be filled up uniformly for the surface of the attractor to be depicted. Figure 3 illustrates the values of d obtained, against N for the series under consideration.

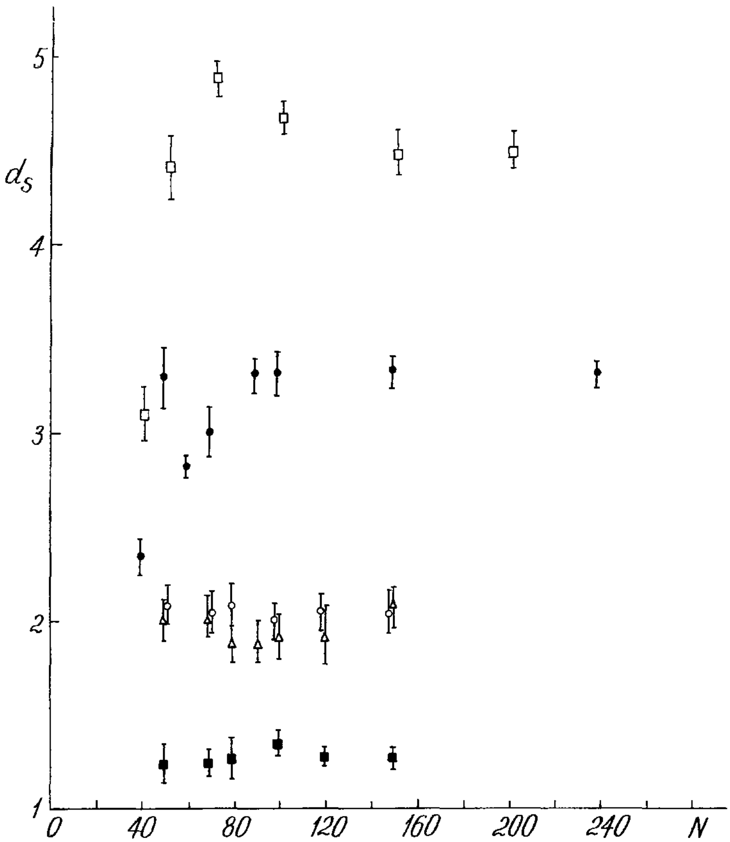

Fig. 3. The influence of experimental points number N on the dimension d_s for the series considered: ● – Wolf numbers averaged over a year (a); □ – radiocarbon data (c); ○ – Lorenz attractor; △ – Rossler attractor; ■ – Henon attractor.

To estimate theoretically $N_{\min}$ we make use of the formula derived by Mayer-Kress (1987):

$$N_{\min} \simeq \left(\frac{S}{\Delta S}\right)^d, \tag{3}$$

where S is an attractor diameter and ΔS is a mean distance between segment trajectories within a phase space. An application of this relationship to the well-known Henon- and Lorenz-type attractors together with their maps gives $N_{\min} \simeq 40$ and $N_{\min} \simeq 80$, respectively. Unfortunately, phase portraits of Wolf numbers and radio-carbon data are not known. Therefore, we suppose $\Delta S \approx \Delta r$ (Mayer-Kress, 1987), Δr being the mean distance between points belonging to the attractor. This value can easily be expressed in terms of a mutual distance distribution function $f(r') \sim (r')^{d-1}$ within a rectilinear segment ($r_{\min} < r' < r_{\max}$):

$$\Delta r \simeq \int_{r_{\min}}^{r_{\max}} r' f(r')\,\mathrm{d}r' \Big/ \int_{r_{\min}}^{r_{\max}} f(r')\,\mathrm{d}r' \simeq \frac{d}{d+1}\, r_{\max}\,, \quad (r_{\max} \gg r_{\min})\,. \tag{4}$$

Thus, for monthly- and yearly-averaged Wolf numbers we get $N_{\min} \simeq 45$ ($r_{\max} \approx 130$, $S \approx 250$, and $d \approx 4.3$) and $N_{\min} \simeq 70$ ($r_{\max} \approx 90$, $S \approx 250$, and $d \approx 3.3$), respectively. An analogous value for radiocarbon data is $N_{\min} \simeq 75$ ($r_{\max} \approx 2$, $S \approx 5$, and $d \approx 4.7$). So, we see a good agreement between theoretically derived values $N_{\min}$ and those from numerical calculations.

Since $N_{\min}$ is rather small one may treat our series by splitting it into pieces in order to find the time fluctuations of d. Such a procedure allows us to describe 'local' properties of the attractor only. The investigation of a long time modulation requires more data and therefore is beyond the scope of our consideration. In Table I we summarize our results for the data arrays within various time periods (Ostryakov and Usoskin, 1988).

TABLE I

C^{14}	1570–1640	1645–1715	1720–1790	1800–1870	1524–1900
d_{SC}	4.3	8.0	4.8	4.6	4.7
W	1749–1987 year-avr.	1792–1828 month-avr.	1848–1859 month-avr.	1749–1771 month-avr.	1848–1859 month-avr. smoothed
d_{SW}	3.3	3.0	4.0	4.3	2.0

One more quantitative parameter which measures the level of chaos is the Kolmogorov–Sinai (or metric) entropy, K. This value is closely connected with new information produced by a nonlinear system if $K > 0$. If we have a time realization of one phase space coordinate only we can use the procedure of Packard *et al.* (1984), to find a lower limit for K. As one can see in Schuster (1984), $K \geq K_2$, where

$$K_2 \approx K_{2s} = \frac{1}{\tau} \ln \frac{C_m^{(s)}(r)}{C_m^{(s+1)}(r)} \tag{5}$$

and

$$C_m^{(s)}(r) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{m} \theta\left(r - \sqrt{\sum_{k=1}^{s} \| \mathbf{X}_{i+k} - \mathbf{X}_{j+k} \|^2}\right). \tag{6}$$

Thus, the positivity of K_2 is a reliable criterion of a stochastisity together with d. According to formula (5) an increase in s results in a decrease in K_{2s} which after reaching the plateau is an accurate estimation of K_{2s}. In Figure 4 is shown an example of these calculations with K_{2s} equal to ~ 0.018 bits month^{-1} for the case of (b) (see Abstract).

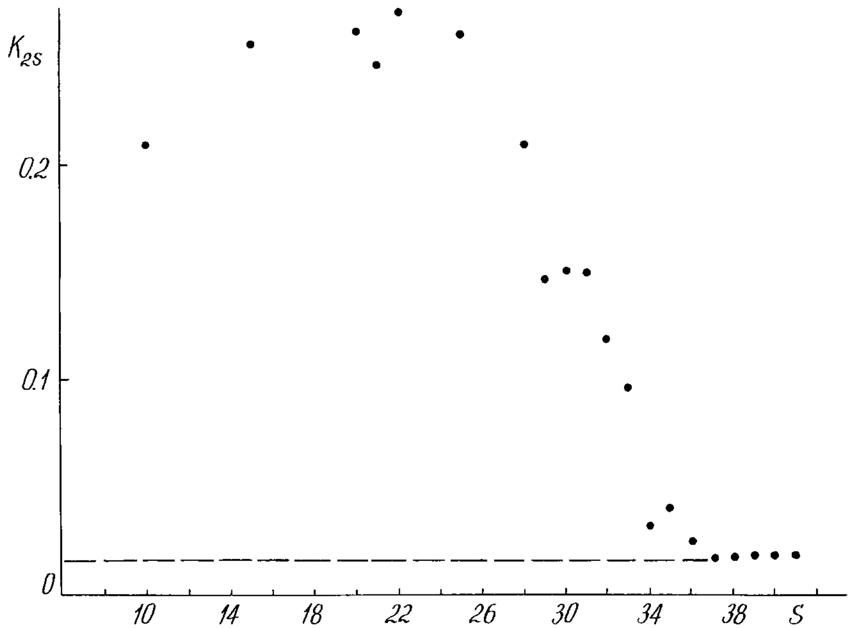

Fig. 4. Kolmogorov entropy for the Wolf numbers averaged over a month (1749–1810, data average (b)); $r = 140$, $n = 13$.

This value obtained for the separate parts of the series (we use four) coincides with that for the whole series (Aimanova and Makarenko, 1988).

The entropy determines the chaotic properties of nonlinear systems as a whole, whereas positive Lyapunov exponents show the rate of creating new information along the principle axis separately. Generally speaking that means that K is a sum of the positive Lyapunov exponents $\lambda_i^{(+)}$:

$$K = \sum_{i=1}^{m} \lambda_i^{(+)} . \tag{7}$$

To obtain a dominant $\lambda_1^{(+)}$ we use a numerical technique developed by Wolf *et al.* (1985). It is $\lambda_1^{(+)} \sim 0.03 - 0.04$ bits month^{-1} (Figure 5).

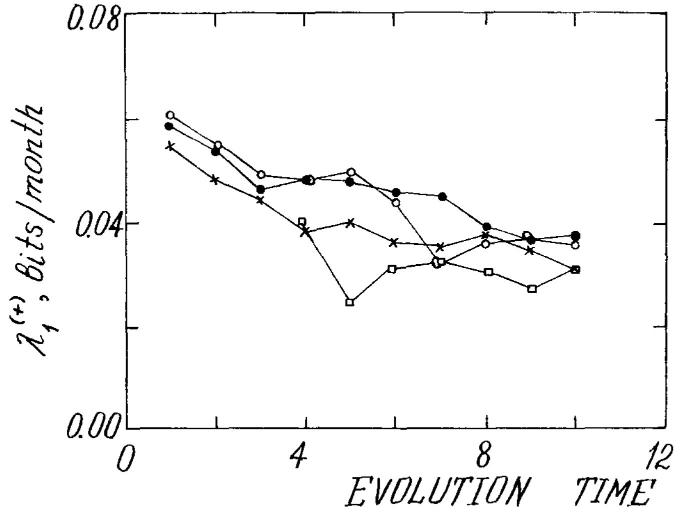

Fig. 5. A dominant value $\lambda_1^{(+)}$ for monthly averaged Wolf numbers (1749–1987) as a function of evolution time (see Wolf *et al.*, 1985) in units of τ: ● $-r = 40, n = 8$; ○ $-r = 60, n = 7$; × $-r = 60, n = 8$; □ $-r = 60, n = 9$.

3. Discussion

The calculated dimension of the attractor which describes solar activity by means of Wolf numbers (a) is $d_{SW} \approx 3.3$ ($n \geq 7$). That is, the main features of the activity can be drawn from using some deterministic set of nonlinear equations with the number of independent variables not less than 4 (see Malinetzkii, Ruzmaikin, and Samarskii, 1986). The analogous value for radiocarbon data turns to be sufficiently higher: $d_{SC} \approx 4.7$ ($n \geq 9$). It becomes clear if one keeps in mind that radiocarbon production rate variations are caused by some other additional factors (except modulation by the Sun): geomagnetic field, solar flares and perhaps climate. It is seen from Table I that there are some peculiarities in the dimensions obtained. They are: the value of d_{SC} during the Maunder minimum dramatically differs from those within neighbouring time intervals and is close to the chaotic one. It seems that such an increase does not result from the sun's behaviour because in contrast, during the period of phase catastrophe (1792–1828), which is similar to the Maunder minimum, d_{SC} becomes smaller: 3.0 (b).

Now we briefly dwell upon the predictability of the solar activity. If we know the state of our system within phase space with the accuracy of Δh then the information we get about that is:

$$H = -\log_2 \Delta h, \quad \text{bit.} \tag{8}$$

Because $K = \mathrm{d}H/\mathrm{d}t \approx H/T$ then the mean time-scale on which our system can not be predicted is given by

$$T = -\frac{1}{K}\log_2 \Delta h\,. \tag{9}$$

It is important that the dependence on Δh here is rather weak (logarithmic). Making use of the lower limit for K from the previous section, one can obtain the upper limit for T. Indeed, taking into account that $\Delta h \sim 25\%$ for Wolf numbers averaged over a month $T \leq 10$ years. The more accurate T estimation can be done with $\lambda_1^{(+)}$ instead of K_{2s}. This yields $T \leq 5$ years. So it would have been impossible to predict solar activity for a period longer than several years even if we had known a set of equations for it. In other words each 11-year cycle fully forgets the previous one.

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