

ON THE CONICAL DENSITY PROPERTIES OF MEASURES ON \mathbb{R}^n

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ABSTRACT. We compare conical density properties and spherical density properties for general Borel measures on \mathbb{R}^n . As consequence, we obtain results for packing and Hausdorff measures \mathcal{P}_h and \mathcal{H}_h provided that the gauge function h satisfies certain conditions.

One consequence of our general results is the following: Let $m, n \in \mathbb{N}$, $0 < s < m \leq n$, $0 < \eta < 1$, and suppose that V is an m -dimensional linear subspace of \mathbb{R}^n . Let μ be either the s -dimensional Hausdorff measure or the s -dimensional packing measure restricted to a set A with $\mu(A) < \infty$. Then for μ -almost every $x \in \mathbb{R}^n$, there is $\theta \in V \cap S^{n-1}$ such that

$$\liminf_{r \downarrow 0} r^{-s} \mu(B(x, r) \cap H(x, \theta, \eta)) = 0,$$

where $H(x, \theta, \eta) = \{y \in \mathbb{R}^n : (y - x) \cdot \theta > \eta|y - x|\}$.

1. INTRODUCTION AND NOTATION

In this paper we study the following question: Suppose that μ is a measure on \mathbb{R}^n and $h:]0, r_0[\rightarrow]0, \infty[$ is a function such that for a typical point x the measures $\mu(B(x, r_i))$ of some small balls $B(x, r_i)$ behave roughly like $h(r_i)$. What can be said about measures on cones?

Let us begin with some notation. Let μ be a measure on \mathbb{R}^n , $A \subset \mathbb{R}^n$, and $h:]0, r_0[\rightarrow]0, \infty[$. The lower and upper μ -densities of the set A at a point $x \in \mathbb{R}^n$ with respect to h are defined by

$$\underline{D}_h(\mu, A, x) = \liminf_{r \downarrow 0} \mu(B(x, r) \cap A) / h(r),$$

$$\overline{D}_h(\mu, A, x) = \limsup_{r \downarrow 0} \mu(B(x, r) \cap A) / h(r),$$

where $B(x, r)$ is the closed ball $B(x, r) = \{y \in \mathbb{R}^n : |y - x| \leq r\}$. Open balls will be denoted by $U(x, r)$. When $A = \mathbb{R}^n$, we abbreviate $\underline{D}_h(\mu, \mathbb{R}^n, x) = \underline{D}_h(\mu, x)$, and similarly with upper densities. Often $h(r) = r^s$ for some $0 \leq s \leq n$. In this

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case we use the notation $\underline{D}_s(\mu, A, x)$ for $\underline{D}_h(\mu, A, x)$ and so on. For example,

$$\underline{D}_s(\mu, x) = \liminf_{r \downarrow 0} \mu(B(x, r)) / r^s.$$

Suppose that $m, n \in \mathbb{N}$ are such that $m \leq n$. The collection of m -dimensional linear subspaces of \mathbb{R}^n is denoted by $G(n, m)$. A natural metric on $G(n, m)$ is given by

$$d(V, W) = \|P_V - P_W\| \text{ for all } V, W \in G(n, m),$$

where P_V is the orthogonal projection onto the subspace V and $\|\cdot\|$ is the usual operator norm for linear mappings. We also use the notation proj_i for the projections onto the coordinate axes. The following identity is sometimes very useful (see [18, pp. 17-20] for a proof):

$$d(V, W) = \sup\{d(x, W) : x \in V \cap S^{n-1}\}, \quad (1.1)$$

where $S^{n-1} = \partial B(0, 1) \subset \mathbb{R}^n$ and on the right hand side $d(y, A)$ stands for the distance from y to A . Let $\gamma_{n,m}$ be the unique Radon probability measure on $G(n, m)$ which is invariant with respect to the orthogonal group $O(n)$, see [16, §3]. Recall that Borel regular and locally finite measures on \mathbb{R}^n are called Radon measures.

Next we introduce some conical objects, see also [16, §11]. Let $x \in \mathbb{R}^n$, $V \in G(n, m)$, $\theta \in S^{n-1}$, $L_\theta = \{t\theta : t > 0\}$, and $0 \leq \eta \leq 1$. We define

$$\begin{aligned} X(x, V, \eta) &= \{y \in \mathbb{R}^n : d(y - x, V) < \eta|y - x|\}, \\ X^+(x, \theta, \eta) &= \{y \in \mathbb{R}^n : d(y - x, L_\theta) < \eta|y - x|\}, \\ H(x, \theta, \eta) &= \{y \in \mathbb{R}^n : (y - x) \cdot \theta > \eta|y - x|\}, \\ H(x, \theta) &= H(x, \theta, 0) = \{y \in \mathbb{R}^n : (y - x) \cdot \theta > 0\}. \end{aligned}$$

One easily verifies the following equality

$$H(x, \theta, \eta) = X^+(x, \theta, (1 - \eta^2)^{1/2}).$$

The H notation is often used for cones that are almost half-spaces, whereas X^+ stands usually for a very narrow cone.

Let $h: [0, r_0[\rightarrow]0, \infty[$ with $h(0) = 0$. Let \mathcal{H}_h be the generalised Hausdorff measure which is constructed using the gauge function h , see [16, §4.9]. If $A \subset \mathbb{R}^n$ has finite \mathcal{H}_h measure, then for \mathcal{H}_h -almost every $x \in A$

$$\overline{D}_h(\mathcal{H}_h, A, x) \leq \limsup_{r \downarrow 0} h(2r) / h(r). \quad (1.2)$$

The corresponding result for the spherical density of the generalised packing measure \mathcal{P}_h is

$$\underline{D}_h(\mathcal{P}_h, A, x) \leq \limsup_{r \downarrow 0} h(2r) / h(r), \quad (1.3)$$

which is true for \mathcal{P}_h -almost every $x \in A$ provided that $\mathcal{P}_h(A) < \infty$. Inequalities (1.2) and (1.3) can be proved with similar arguments as in the familiar case

$h(r) = r^s$, see [16, theorems 6.2, 6.10]. The definition of \mathcal{P}_h can be found for example in [3, definition 2.2]. When $h(r) = r^s$ for some $0 \leq s \leq n$, the common notations \mathcal{H}^s and \mathcal{P}^s are used for \mathcal{H}_h and \mathcal{P}_h , respectively.

We will now discuss some conical density results that are known for Hausdorff measures. We begin with the following theorem on lower densities. If μ is a measure on \mathbb{R}^n and $A \subset \mathbb{R}^n$, the notation $\mu \llcorner A$ stands for the restriction measure, that is $\mu \llcorner A(B) = \mu(A \cap B)$ for all $B \subset \mathbb{R}^n$.

Theorem 1.1. *Assume that $0 < s < n$ and $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$.*

- (1) *If $\eta > 0$, then for \mathcal{H}^s -almost all $x \in A$, there exists $\theta = \theta(x) \in S^{n-1}$ such that $\underline{D}_s(\mathcal{H}^s \llcorner A, H(x, \theta, \eta), x) = 0$.*
- (2) *If $0 < s < 1$ and $\theta \in S^{n-1}$, then $\underline{D}_s(\mathcal{H}^s \llcorner A, H(x, \theta), x) = 0$ for \mathcal{H}^s -almost all $x \in A$.*
- (3) *If $n - 1 < s < n$, then (1) holds also with the value $\eta = 0$.*

Marstrand [10, pp. 293–297] proved Theorem 1.1 in \mathbb{R}^2 and his method can be generalised to higher dimensions, see also [4, pp. 56–61]. Besicovitch [1, theorem 2] had earlier proved (2) in \mathbb{R} . He has also shown [2, theorem 13] that if $A \subset \mathbb{R}^2$ is purely 1-unrectifiable and $\theta \in S^1$, then $\underline{D}_1(\mathcal{H}^1 \llcorner A, H(x, \theta), x) = 0$ for \mathcal{H}^1 -almost all $x \in A$. Gillis [7] had earlier proved this for cones $H(x, \theta, \eta)$, when $\eta > 0$. Mattila [13] studied h -densities of singular measures in \mathbb{R} and obtained as a corollary a result somewhat similar to (2) [13, corollary 9].

Theorem 1.1 (1) does not hold for all $\theta \in S^{n-1}$, see example 2.5. Recently Lorent [9] showed that one can choose directions θ in (1) to lie on a fixed $(n - 1)$ -dimensional linear subspace of \mathbb{R}^n provided that either $s = n - 1$ and A is purely $(n - 1)$ -unrectifiable or $s < n - 1$. In Section 2 we shall proceed in this direction by showing that for all $m \in \mathbb{N}$, $m \leq n$, the following is true: If $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$ and either $s = m$ and A is purely m -unrectifiable or $s < m$, then the directions θ in (1) can be chosen to lie on a fixed m -dimensional linear subspace. We will also prove the corresponding result for the s -dimensional packing measure \mathcal{P}^s when $0 < s < m$. Moreover, our results can be applied to many other measures with the property $\underline{D}_h(\mu, x) < \infty$ for μ -almost all $x \in \mathbb{R}^n$, see Theorems 2.1 and 2.2. One could also modify Marstrand's methods from [10] to prove claims (1) and (2) of Theorem 1.1 for measures \mathcal{P}^s .

Concerning upper conical densities for Hausdorff measures the fundamental theorem is as follows:

Theorem 1.2. *Suppose that $m \in \mathbb{N}$, $m < s < n$, $0 < \eta \leq 1$, and $A \subset \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$.*

- (1) *If $V \in G(n, n - m)$, then*

$$\overline{D}_s(\mathcal{H}^s \llcorner A, X(x, V, \eta), x) > c = c(n, m, \eta) > 0$$

for \mathcal{H}^s -almost all $x \in A$.

(2) If $m = n - 1$ and $\theta \in S^{n-1}$, then

$$\overline{D}_s(\mathcal{H}^s \llcorner A, X^+(x, \theta, \eta), x) > c = c(n, s, \eta) > 0$$

for \mathcal{H}^s -almost all $x \in A$.

The preceding theorem is also due to Marstrand [10, theorem IX] in \mathbb{R}^2 . Salli [18] generalised Marstrand's result to \mathbb{R}^n and proved also a corresponding result for cones generated by open sets $G \subset G(n, n - m)$ or $S \subset S^{n-1}$. Mattila [14, theorem 3.3] went even further by proving a very general upper density theorem for Hausdorff measures. Besicovitch [1, theorem 2] has shown that if $0 < s < 1$ and $A \subset \mathbb{R}$ is \mathcal{H}^s measurable with $\mathcal{H}^s(A) < \infty$, then $\overline{D}_s(\mathcal{H}^s \llcorner A,]x, \infty[, x) = \overline{D}_s(\mathcal{H}^s \llcorner A,]-\infty, x[, x) = 1$ for \mathcal{H}^s -almost all $x \in A$.

There are also upper conical density results for purely m -unrectifiable sets. Besicovitch showed in [2, theorem 8] that if $l \in G(2, 1)$, $0 < \eta < 1$, and $A \subset \mathbb{R}^2$ is purely 1-unrectifiable, then $\overline{D}_1(\mathcal{H}^1 \llcorner A, X(x, l, \eta), x) > c(\eta) > 0$ for \mathcal{H}^1 almost every $x \in \mathbb{R}^2$. He also gave an example [2, p. 327–328] to illustrate that the above assertion is not true for the one sided cones $X^+(x, \theta, \eta)$. Federer [6, theorem 3.3.17] gave a generalisation of the above result. See also [16, corollary 15.16].

In Section 3 we shall discuss how Theorem 1.2 can be generalised for packing measures, or more generally, for measures μ such that $\underline{D}_h(\mu, x)$ is finite μ -almost everywhere, with a suitable function h .

The above mentioned conical density theorems have been used in various different ways. Let us briefly explain some of them. Marstrand ([10], [11], [12]) used conical density arguments to prove the fairly deep fact that for non-integral values of s there are no Radon measures μ so that the positive and finite density, $\lim_{r \downarrow 0} \mu(B(x, r)) / r^s$, would exist in a set of positive μ measure. Mattila [14] used his upper density result to find an upper bound for the dimension of strongly porous sets. Theorem 1.1 (1) and Lorent's [9] generalisation of it can also be used to give some light on the problem of characterising removable sets for harmonic functions, see [17] and [9]. Note also that if $\mathcal{H}^s(A) < \infty$ and $\underline{D}_s(\mathcal{H}^s, A, x) > 0$ almost everywhere, then Theorem 1.1 (1) is equivalent to the following statement: If $\mu = \mathcal{H}^s \llcorner A$, then for μ -almost every x , there is a tangent measure of μ at x which is supported on one side of some $(n - 1)$ -plane $V \in G(n, n - 1)$. This property is sometimes very useful, see for example [16].

Throughout this paper we shall deal only with Borel measures. This is only for convenience, as it is readily seen that our results remain true for general measures.

2. LOWER DENSITIES

The main result of this section is the following theorem. It generalises the theorem of Lorent [9, theorem 1] in several different ways: Instead of r^s we can have more general density functions and instead of only $(n - 1)$ -planes we can have m -planes for any integer $0 < m < n$. Maybe the most important

improvement, however, is that we only need to assume $\underline{D}_h(\mu, x) < \infty$ and not $\overline{D}_h(\mu, x) < \infty$. Lorent's method might work for measures with $\overline{D}_h(\mu, x) < \infty$, but also in this case our proof is simpler. In case $\underline{D}_h(\mu, x) < \infty$ the method of Lorent would not work, but the idea of using projection arguments in our proof was gotten from his proof.

Theorem 2.1. *Let $m, n \in \mathbb{N}$ with $m \leq n$. Assume that $h:]0, r_0[\rightarrow]0, \infty[$ fulfils the following three conditions:*

$$\lim_{r \downarrow 0} h(r) = 0, \quad (\text{h1})$$

$$\lim_{r \downarrow 0} h(r) / r^m = \infty, \quad (\text{h2})$$

$$h(r_1) + h(r_2) \geq h\left((r_1^m + r_2^m)^{1/m}\right) \text{ whenever } r_1^m + r_2^m \leq r_0^m. \quad (\text{h3})$$

Suppose that $V \in G(n, m)$, $\eta > 0$, and μ is a Borel measure on \mathbb{R}^n with $\underline{D}_h(\mu, x) < \infty$ for μ -almost all $x \in \mathbb{R}^n$. Then for μ -almost all $x \in \mathbb{R}^n$, there is $\theta = \theta(x) \in V \cap S^{n-1}$ such that $\underline{D}_h(\mu, H(x, \theta, \eta), x) = 0$.

If above $m = 1$, then we can say a little bit more. The following theorem should be compared with Theorem 1.1 (2).

Theorem 2.2. *Assume that $h:]0, r_0[\rightarrow]0, \infty[$ fulfils the assumptions (h1)–(h3) of Theorem 2.1 with $m = 1$. If $\theta \in S^{n-1}$ and μ is a Borel measure on \mathbb{R}^n such that $\underline{D}_h(\mu, x) < \infty$ for μ -almost all $x \in \mathbb{R}^n$, then $\underline{D}_h(\mu, H(x, \theta), x) = 0$ for μ -almost all $x \in \mathbb{R}^n$.*

Before the proofs, let us make some remarks concerning Theorems 2.1 and 2.2

If $0 < s < m$, $h(r) = r^s$, and $\mu = \mathcal{H}^s \llcorner A$, where A is a subset of \mathbb{R}^n with $\mathcal{H}^s(A) < \infty$, then the assumptions of Theorem 2.1 are clearly satisfied, recall (1.2). If in addition $s < 1$, then the assumptions of Theorem 2.2 are valid. Above one can also replace Hausdorff measures by packing measures. Hence we have:

Corollary 2.3. *Let μ be either \mathcal{H}^s or \mathcal{P}^s , and let $A \subset \mathbb{R}^n$ with $\mu(A) < \infty$.*

- (1) *If $\eta > 0$, $0 < s < m$, and $V \in G(n, m)$, then for μ -almost all $x \in A$, there is $\theta = \theta(x) \in V \cap S^{n-1}$ such that $\underline{D}_s(\mu \llcorner A, H(x, \theta, \eta), x) = 0$.*
- (2) *If $0 < s < 1$ and $\theta \in S^{n-1}$, then $\underline{D}_s(\mu \llcorner A, H(x, \theta), x) = 0$ for μ -almost all $x \in A$.*

There are also many other functions than r^s to which Theorems 2.1 and 2.2 can be applied. For example $h(r) = r^s \log(1/r)$ fulfils (h1)–(h3) when $0 < s \leq m$. Also, if h is differentiable with non-increasing derivative, and satisfies (h1), then (h2) and (h3) are valid with $m = 1$. Using density bounds (1.2) and (1.3), we see that the statements of Corollary 2.3 remain true if we let μ to be \mathcal{H}_h or \mathcal{P}_h provided that $h(0) = 0$ and that h fulfils (h1)–(h3) (with $m = 1$ in (2)).

If in Theorem 2.1 or 2.2 also $\underline{D}_h(\mu, x) > 0$ for μ -almost every x , then the densities $\underline{D}_h(\mu, H(x, \theta, \eta), x)$ can be replaced by $\underline{D}_\mu(H(x, \theta, \eta), x)$, where

$$\underline{D}_\mu(H(x, \theta, \eta), x) = \liminf_{r \downarrow 0} \mu(H(x, \theta, \eta) \cap B(x, r)) / \mu(B(x, r))$$

is the conical density with respect to μ .

The assertion (3) of Theorem 1.1 follows from Theorem 1.1 (1) together with the following observation: If $n - 1 < s < n$, $0 < \mathcal{H}^s(A) < \infty$, and $\theta \in S^{n-1}$, then (1.2) implies that

$$\lim_{\eta \downarrow 0} \overline{D}_s(\mathcal{H}^s \llcorner A, H(x, \theta) \setminus H(x, \theta, \eta), x) = 0 \quad (2.1)$$

for \mathcal{H}^s -almost all $x \in A$. It remains open, if assertion (3) of Theorem 1.1 is valid for packing measures. However, if we know that the assumptions of Theorem 2.1 hold, $\overline{D}_h(\mu, x) < \infty$ for μ -almost every $x \in \mathbb{R}^n$, and

$$\liminf_{\alpha \downarrow 0} \limsup_{r \downarrow 0} \alpha^{1-n} h(\alpha r) / h(r) = 0, \quad (2.2)$$

then we can prove, with the help of Theorem 2.1 and a generalisation of (2.1), that for μ -almost every $x \in \mathbb{R}^n$, there is $\theta \in S^{n-1}$ so that $\underline{D}_h(\mu, H(x, \theta), x) = 0$. The condition (2.2) is valid for example if $n - 1 < s < n$ and $h(r) = r^s \log(1/r)$.

Condition (h3) may be weakened a little bit. It suffices to assume that there is a constant $0 < c \leq 1$ such that

$$\sum_{i=1}^{\infty} h(r_i) \geq c h \left(\left(\sum_{i=1}^{\infty} r_i^m \right)^{1/m} \right) \text{ whenever } \sum_{i=1}^{\infty} r_i^m \leq r_0^m. \quad (2.3)$$

This is immediate corollary to the following lemma.

Lemma 2.4. *If $h:]0, r_0[\rightarrow]0, \infty[$ fulfils inequality (2.3), then there is a function \tilde{h} that satisfies (h3) and inequalities $ch \leq \tilde{h} \leq h$.*

Proof. Define $\tilde{h}(r) = \inf\{\sum_i h(r_i) : \sum_i r_i^m = r^m\}$. Now the desired properties clearly hold for \tilde{h} . \square

Assumptions (h1) and (h2) for the function h in Theorems 2.1 and 2.2 are well justified but it is natural to ask whether assumption (h3), or (2.3), is really needed. In example 2.12 we shall construct a Radon measure μ and a function h which show that assumptions (h1) and (h2) alone are not enough to guarantee the assertions of Theorems 2.1 and 2.2. Of course, this does not exclude the possibility that assumption (h3) could be weakened and as P. Mattila pointed out to me, it is an interesting question whether it can be replaced by a doubling condition on h , see Theorem 3.1.

Theorem 2.1 is in a sense sharp for Hausdorff measures. This is shown by the following simple example.

Example 2.5. The assertion of Theorem 2.1 is not valid for measures $\mathcal{H}^s \llcorner A$, where $\mathcal{H}^s(A) < \infty$, and $m \leq s \leq n$: Consider a Cantor set $C \subset \mathbb{R}^{n-m}$ such that $0 < \mathcal{H}^{s-m}(C) < \infty$. Let $A = (U(0, 1) \cap \mathbb{R}^m) \times C \subset \mathbb{R}^n$, $\mu = \mathcal{H}^s \llcorner A$, and $0 \leq \eta < 1$. Then $\underline{D}_s(\mu, H(x, \theta, \eta), x) > 0$ for all $x \in A$, and $\theta \in V \cap S^{n-1}$, where $V = \{x \in \mathbb{R}^n : \text{proj}_i x = 0 \text{ for all } i = m+1, \dots, n\} \in G(n, m)$.

The following simple lemma is essential in our proofs.

Lemma 2.6. *Assume that μ is a Borel measure on \mathbb{R}^n , $A \subset \mathbb{R}^n$ is a Borel set, and $h:]0, r_0[\rightarrow]0, \infty[$. If for μ -almost every $x \in A$, there is $r > 0$ such that $\mu(B(x, r)) < \infty$, then for μ -almost all $x \in A$*

- (1) $\underline{D}_h(\mu, A, x) = \underline{D}_h(\mu, x)$,
- (2) $\overline{D}_h(\mu, A, x) = \overline{D}_h(\mu, x)$.

Proof. We prove (1). Claim (2) can be established similarly. Because μ -almost all points of A are contained in a countable union of open balls, each of finite μ measure, we may assume that μ is finite.

Clearly $\underline{D}_h(\mu, A, x) \leq \underline{D}_h(\mu, x)$ for all $x \in A$. It is well known (see [16, corollary 2.14], for example) that

$$\lim_{r \downarrow 0} \mu(B(x, r) \cap A) / \mu(B(x, r)) = 1$$

for μ -almost every $x \in A$. Take such a point x and fix radii $r_i \downarrow 0$ such that

$$\mu(B(x, r_i) \cap A) / h(r_i) \longrightarrow \underline{D}_h(\mu, A, x)$$

as $i \rightarrow \infty$. Now

$$\frac{\mu(B(x, r_i))}{h(r_i)} = \frac{\mu(B(x, r_i) \cap A)}{h(r_i)} \frac{\mu(B(x, r_i))}{\mu(B(x, r_i) \cap A)} \longrightarrow \underline{D}_h(\mu, A, x)$$

as $i \rightarrow \infty$ and thus $\underline{D}_h(\mu, A, x) \geq \underline{D}_h(\mu, x)$. \square

Proof of Theorem 2.2. We may assume that $\theta = e_1 = (1, 0, \dots, 0)$. Let $\alpha, \beta \in]0, \infty[$ and define

$$A = \{x \in \mathbb{R}^n : \mu(B(x, r) \cap H(x, \theta)) \geq \alpha h(r) \text{ for all } 0 < r < \beta\}. \quad (2.4)$$

We will begin by showing that A is a Borel set. Let us state this as a lemma for later use.

Lemma 2.7. *A is a Borel set.*

Proof of Lemma 2.7. Using assumptions (h1) and (h3), it is easily seen that if $0 < r < r_0$ and $q_i \uparrow r$ as $i \rightarrow \infty$, then

$$\liminf_{i \rightarrow \infty} h(q_i) \geq h(r).$$

Using this one obtains

$$A = \bigcap_{\substack{0 < q < \beta \\ q \in \mathbb{Q}}} \{x \in \mathbb{R}^n : \mu(U(x, q) \cap H(x, \theta)) \geq \alpha h(q)\}.$$

Fix $q > 0$. It remains to show that the mapping

$$f : x \mapsto \mu(U(x, q) \cap H(x, \theta))$$

is lower semicontinuous. Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$\mu(\{y \in U(x, q) \cap H(x, \theta) : d(y, \partial(U(x, q) \cap H(x, \theta))) > \delta\}) > f(x) - \varepsilon,$$

giving $f(y) > f(x) - \varepsilon$ for all $|y - x| < \delta$. \square

We continue to prove Theorem 2.2. Suppose that $F \subset A$ is closed. It suffices to show that $\mu(F) = 0$. Assume on the contrary that $\mu(F) > 0$. By Lemma 2.6 (1), we find $x \in F$ such that $\underline{D}_h(\mu, F, x) = \underline{D}_h(\mu, x) = c < \infty$. So we can find small radii r such that $\mu(B(x, r)) \approx \mu(B(x, r) \cap F) \approx ch(r)$. To be exact, let us fix $\varepsilon < \alpha/(8c + 4)$ and $0 < r < r_0$ such that $\mu(B(x, 2r)) < \infty$ (recall $\underline{D}_h(\mu, x) < \infty$) and

$$(1 - \varepsilon)ch(r) < \mu(B(x, r) \cap F) \leq \mu(B(x, r)) < (1 + \varepsilon)ch(r),$$

whence also

$$\mu(B(x, r) \setminus F) < 2c\varepsilon h(r). \quad (2.5)$$

Let $\gamma > 0$. According to assumption (h2), there is $\eta > 0$ such that $r < \gamma h(r)$, whenever $0 < r < \eta$. We can now apply Vitali covering theorem [6, corollary 2.8.15] to the collection

$$\{B(z, \rho) : z \in F \cap B(x, r), \rho < \min\{r, \beta, \eta\}\}$$

to find disjoint balls $B_i = B(z_i, r_i)$ for $i = 1, \dots, N$ from this collection such that

$$\mu\left((B(x, r) \cap F) \setminus \bigcup_{i=1}^N B_i\right) < \varepsilon h(r). \quad (2.6)$$

We may assume that $x \in \bigcup_{i=1}^N B_i$. Since $\mu(B(x, 2r)) < \infty$, reducing γ enough, and using (2.4), we are led to the estimate

$$\begin{aligned} \sum_{i=1}^N r_i &< \gamma \sum_{i=1}^N h(r_i) \leq \gamma \sum_{i=1}^N \mu(B_i \cap H(z_i, \theta)) / \alpha \leq \gamma \mu(B(x, 2r)) / \alpha \\ &< r/8. \end{aligned} \quad (2.7)$$

We now choose points $x_i \in B_i \cap F$ for $i = 1, \dots, N$ such that

$$\text{proj}_1(x_i) = \max\{\text{proj}_1(y) : y \in B_i \cap F\}.$$

Let U_i denote the open ball that has double radius but same centre as B_i . Next we select recursively disjoint points $y_1, \dots, y_l \in \{x_i\}_{i=1}^N$ and radii d_1, \dots, d_l for some $l \leq N$ as follows (see figure 1 to get some idea of the construction): Take $y_1 \in \{x_i\}_{i=1}^N$ such that $\text{proj}_1(y_1) = \max_{i=1, \dots, N} \text{proj}_1(x_i)$, and define $d_1 = \max\{0, r - |x - y_1|\}$. If points y_1, \dots, y_k and radii d_1, \dots, d_k have been selected, let $0 < a_k \leq |x - y_k| - d_k$ be the greatest radius such that $\partial B(x, a_k) \cap \bigcup_{i=1}^N U_i = \emptyset$ (If such an a_k does not exist, then we finish our selection). We now

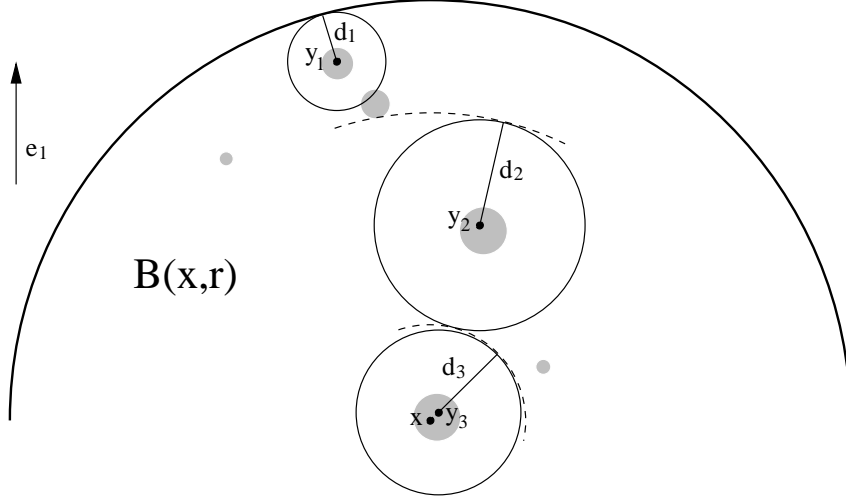


FIGURE 1. Selection of points y_i and radii d_i . Above the grey balls are the balls U_i .

choose $y_{k+1} \in B(x, a_k) \cap \{x_i\}_{i=1}^N$ such that $\text{proj}_1(y_{k+1}) = \max\{\text{proj}_1(x_i) : i = 1, \dots, N \text{ and } x_i \in B(x, a_k)\}$ and define $d_{k+1} = a_k - |x - y_{k+1}|$. When this selection terminates, we have defined points y_1, \dots, y_l and radii d_1, \dots, d_l for some $l \leq N$. As a result of the above construction, and with the help of (2.7), we get

$$r \leq \sum_{i=1}^N 4r_i + \sum_{i=1}^l 2d_i \leq r/2 + 2 \sum_{i=1}^l d_i.$$

Therefore, we may make numbers d_i small enough to satisfy

$$\sum_{i=1}^l d_i = r/4. \quad (2.8)$$

It also follows from the construction that for all $i = 1, \dots, l$,

$$B(y_i, d_i) \cap H(y_i, \theta) \cap F \cap \bigcup_{j=i}^N B_j = \emptyset. \quad (2.9)$$

We are now ready to estimate the measure of $B(x, r) \setminus F$. Using the fact that the half balls $B(y_i, d_i) \cap H(y_i, \theta) \subset B(x, r)$ are disjoint, (2.9), (2.4), (2.6), (2.8),

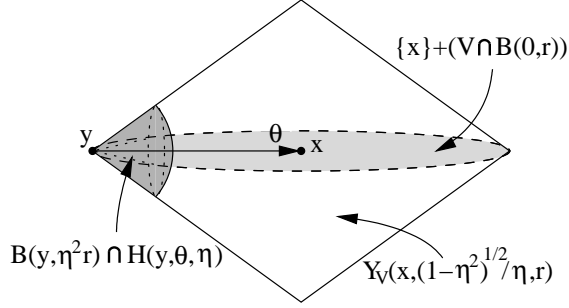


FIGURE 2. See lemma 2.8.

(h3), and the definition of ε , we deduce

$$\begin{aligned}
& \mu(B(x, r) \setminus F) \\
& \geq \sum_{i=1}^l \mu(B(y_i, d_i) \cap H(y_i, \theta)) - \mu\left(B(x, r) \cap F \setminus \bigcup_{i=1}^N B_i\right) \\
& > \alpha \sum_{i=1}^l h(d_i) - \varepsilon h(r) \geq \alpha h(r/4) - \varepsilon h(r) \\
& \geq (\alpha/4 - \varepsilon) h(r) > 2c\varepsilon h(r).
\end{aligned}$$

This contradicts (2.5). \square

Several geometrical lemmas are needed for the proof of Theorem 2.1. Before them, let us define one more conical object. If $V \in G(n, m)$, $x \in \mathbb{R}^n$, $\alpha > 0$, and $r > 0$, then

$$Y_V(x, \alpha, r) = \{y \in \mathbb{R}^n : |P_{V^\perp}(y - x)| \leq \alpha(r - |P_V(y - x)|)\}$$

It follows readily from the above definition that if $y \in Y_V(x, \alpha, r)$, then

$$Y_V(x + P_{V^\perp}(y - x), \alpha, |P_V(y - x)|) \subset Y_V(x, \alpha, r), \quad (2.10)$$

and if $y \in \{x\} + V$ with $|y - x| \leq r$, then

$$Y_V(y, \alpha, r - |y - x|) \subset Y_V(x, \alpha, r). \quad (2.11)$$

See figure 2 for the following lemma.

Lemma 2.8. *Let $0 < \eta < 1$, $r > 0$, $V \in G(n, m)$, $x \in \mathbb{R}^n$, $y \in \{x\} + (\partial B(0, r) \cap V)$, and $\theta = (x - y)/|x - y|$, then*

$$B(y, \eta^2 r) \cap H(y, \theta, \eta) \subset Y_V\left(x, (1 - \eta^2)^{1/2}/\eta, r\right).$$

Proof. In this proof we denote $\text{proj}_i(z)$ by z_i . We may assume without loss of generality that $r = 1$, $y = 0$, $x = e_1$, and also that $V = \{z : z_i = 0 \text{ for all } i = m + 1, \dots, n\}$.

Fix $z \in B(0, \eta^2) \cap H(0, \theta, \eta)$. Then $z \cdot \theta = z_1 > \eta|z|$, and it follows that $z_1^2(1 - \eta^2)/\eta^2 > \sum_{i=2}^n z_i^2$. Using this, we deduce

$$|P_{V^\perp}(z - x)| = \left(\sum_{i=m+1}^n z_i^2 \right)^{1/2} < \left(z_1^2(1 - \eta^2)/\eta^2 - \sum_{i=2}^m z_i^2 \right)^{1/2}. \quad (2.12)$$

We also have

$$r - |P_V(z - x)| = 1 - \left((1 - z_1)^2 + \sum_{i=2}^m z_i^2 \right)^{1/2}. \quad (2.13)$$

The assertion follows from (2.12) and (2.13), if we show that

$$\left(z_1^2 \frac{1 - \eta^2}{\eta^2} - \sum_{i=2}^m z_i^2 \right)^{1/2} \leq \frac{(1 - \eta^2)^{1/2}}{\eta} \left(1 - \left((1 - z_1)^2 + \sum_{i=2}^m z_i^2 \right)^{1/2} \right). \quad (2.14)$$

The proof of (2.14) is elementary calculation, but we give some details for convenience. Note first that $z_1^2(1 - \eta^2)/\eta^2 - \sum_{i=2}^m z_i^2 > 0$ by (2.12) and also $(1 - z_1)^2 + \sum_{i=2}^m z_i^2 < 1$ since $\sum_{i=1}^m z_i^2 \leq |z|^2 \leq |z|\eta^2 < |z|\eta < z_1$ gives $(1 - z_1)^2 + \sum_{i=2}^m z_i^2 < 1 - z_1 + z_1^2$, and $0 < z_1 < 1$. Multiplying both sides of (2.14) by $\eta/(1 - \eta^2)^{1/2}$, taking squares, and reordering terms, it reduces to

$$2 \left((1 - z_1)^2 + \sum_{i=2}^m z_i^2 \right)^{1/2} \leq 2(1 - z_1) + (1 - \eta^2)^{-1} \sum_{i=2}^m z_i^2.$$

Taking squares again, we see that this is equivalent to

$$4 \sum_{i=2}^m z_i^2 \leq \left(4(1 - z_1)/(1 - \eta^2) + (1 - \eta^2)^{-2} \sum_{i=2}^m z_i^2 \right) \sum_{i=2}^m z_i^2.$$

The above inequality is clearly true, since $|z| \leq \eta^2$ implies $1 - z_1 \geq 1 - \eta^2$. \square

Lemma 2.9. *Suppose that $V \in G(n, m)$, $F \subset \mathbb{R}^n$ is a closed set, and $x \in F$ such that $\mathcal{H}^m(P_V(F \cap Y_V(x, \alpha, r))) = 0$. Then there is a disjoint collection $\{Y_i = Y_V(x_i, \alpha, r_i)\}_i$ such that $\sum_i r_i^m \geq 10^{-m} r^m$, and $Y_i \subset Y_V(x, \alpha, r)$ for all i . Furthermore, cones Y_i fulfil:*

$$F \cap \{z \in Y_i : |P_V(z - x_i)| < r_i\} = \emptyset, \quad (2.15)$$

$$F \cap \{z \in Y_i : |P_V(z - x_i)| = r_i\} \neq \emptyset. \quad (2.16)$$

Proof. We may assume without loss of generality that $x \in V$. We will prove that if $y \in (V \cap B(x, r/2)) \setminus P_V(F \cap Y_V(x, \alpha, r))$, then there is a cone $Y_V(z, \alpha, \delta) \subset Y_V(x, \alpha, r)$ which satisfies the conditions (2.15) and (2.16) and for which $y \in P_V(Y_V(z, \alpha, \delta))$. The assertion follows then from this by applying the 5r-covering theorem [16, theorem 2.1] to the projections $P_V(Y_V(z, \alpha, \delta))$.

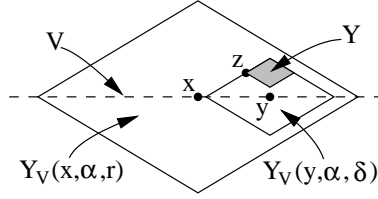


FIGURE 3. See lemma 2.9.

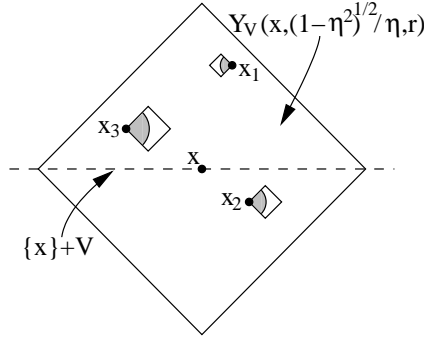


FIGURE 4. The setting of lemma 2.10. The bigger cones are found by lemma 2.9 and the smaller ones (the grey ones) inside them are given by lemma 2.8.

Fix $y \in (V \cap B(x, r/2)) \setminus P_V(F \cap Y_V(x, \alpha, r))$. Then there exists a radius $0 < \delta < |y - x|$ such that $F \cap \text{int } Y_V(y, \alpha, \delta) = \emptyset$ but $F \cap \partial Y_V(y, \alpha, \delta) \neq \emptyset$. Take $z \in F \cap \partial Y_V(y, \alpha, \delta)$ such that

$$|P_V(z - y)| = \min\{|P_V(v - y)| : v \in F \cap \partial Y_V(y, \alpha, \delta)\}. \quad (2.17)$$

Let (see figure 3)

$$Y = Y_V(y + P_{V^\perp}(z - y), \alpha, |P_V(z - y)|).$$

Clearly $y \in P_V(Y)$. By (2.10), and (2.11), we get

$$Y \subset Y_V(y, \alpha, \delta) \subset Y_V(x, \alpha, r).$$

Now (2.17) implies that condition (2.15) holds for Y . Also condition (2.16) is valid for Y , since $z \in F \cap \{w \in Y : |P_V(w - (y + P_{V^\perp}(z - y)))| = |P_V(z - y)|\}$. This completes the proof. \square

The next lemma follows directly from Lemmas 2.9 and 2.8, see figure 4.

Lemma 2.10. *Let $V \in G(n, m)$, and $0 < \eta < 1$. Suppose that $F \subset \mathbb{R}^n$ is a closed set, and $x \in F$ such that*

$$\mathcal{H}^m \left(P_V \left(F \cap Y_V \left(x, (1 - \eta^2)^{1/2} / \eta, r \right) \right) \right) = 0.$$

Then there are disjoint cones

$$B(x_i, r_i) \cap H(x_i, \theta_i, \eta) \subset Y_V \left(x, (1 - \eta^2)^{1/2} / \eta, r \right) \setminus F$$

such that $x_i \in F$, $\theta_i \in V \cap S^{n-1}$ for all i and $\sum_i r_i^m \geq 10^{-m} \eta^{2m} r^m$.

Proof of Theorem 2.1. We may assume, as in Lemma 2.6, that μ is finite. We shall first prove that for all numbers $\alpha, \eta \in]0, 1[$, and $0 < \beta < r_0$, the set

$$A = \{x : \mu(B(x, r) \cap H(x, \theta, \eta)) \geq \alpha h(r) \text{ whenever } 0 < r < \beta \text{ and } \theta \in V \cap S^{n-1}\}$$

is of μ measure zero and then show how this implies our theorem.

By modifying the proof of Lemma 2.7, it is rather easy to see that A is a Borel set. Hence it is sufficient to show that if $F \subset A$ is closed, then $\mu(F) = 0$. Assume on the contrary that $\mu(F) > 0$. Using Lemma 2.6 (1), we find $x \in F$ such that $\underline{D}_h(\mu, F, x) = \underline{D}_h(\mu, x) = c < \infty$. For technical reasons, we assume that $10^{-1} \eta^3 = 2^{-k/m}$ for some $k \in \mathbb{N}$. By iteration of assumption (h3), we obtain a constant $c' > 0$ that depends only on η and m (one can take $c' = 2^{-k}$) such that

$$h(10^{-1} \eta^3 r) \geq c' h(r) \quad (2.18)$$

whenever $r < r_0$. Let $0 < \varepsilon < \alpha c' / (2c)$. As in the proof of theorem 2.1, we find $0 < r < \beta$ such that

$$c(1 - \varepsilon) h(r) < \mu(B(x, r) \cap F) \leq \mu(B(x, r)) < c(1 + \varepsilon) h(r).$$

Then also

$$\mu(B(x, r) \setminus F) < 2\varepsilon c h(r). \quad (2.19)$$

Our next step is to prove that

$$\mathcal{H}^m(B(x, r) \cap F) = 0. \quad (2.20)$$

Let $\gamma > 0$. Using (h2), and the $5r$ -covering theorem [16, theorem 2.1], it is possible to select a collection of disjoint balls,

$$\{B_i = B(x_i, r_i) : x_i \in B(x, r) \cap F \text{ and } r_i^m < \min\{\beta^m, \gamma h(r_i) / 5^m\}\},$$

such that $B(x, r) \cap F \subset \bigcup_i B(x_i, 5r_i)$. We obtain

$$\sum_i (5r_i)^m < \gamma \sum_i h(r_i) < \gamma \sum_i \mu(B_i) / \alpha < \gamma \mu(B(x, r + 1)) / \alpha.$$

Since the right hand side of the above inequality tends to zero as $\gamma \downarrow 0$, this yields (2.20).

Since projections cannot increase Hausdorff measure, we observe that

$$\mathcal{H}^m(P_V(B(x, r) \cap F)) = 0. \quad (2.21)$$

Using the inclusion

$$Y_V \left(x, (1 - \eta^2)^{1/2} / \eta, \eta r \right) \subset B(x, r)$$

combined with Lemma 2.10, we find disjoint cones $B(x_i, r_i) \cap H(x_i, \theta_i, \eta) \subset B(x, r) \setminus F$ such that $x_i \in F$, $\theta_i \in V$ for all i , and furthermore

$$\sum_i r_i^m = 10^{-m} \eta^{3m} r^m.$$

Using the definition of A , the above fact combined with (h3), (2.18), and our choice of ε , we get

$$\begin{aligned} \mu(B(x, r) \setminus F) &\geq \sum_{i=1}^{\infty} \mu(B(x_i, r_i) \cap H(x_i, \theta_i, \eta)) \\ &\geq \alpha \sum_{i=1}^{\infty} h(r_i) \geq \alpha h(10^{-1} \eta^3 r) \\ &\geq \alpha c' h(r) > 2\varepsilon c h(r). \end{aligned}$$

This contradicts (2.19) and thus $\mu(A) = 0$.

Fix $\eta > 0$ and numbers $\alpha_i > 0$ such that $\alpha_i \downarrow 0$ as $i \rightarrow \infty$. We can now find for μ -almost every $x \in \mathbb{R}^n$ directions $\theta_i \in V \cap S^{n-1}$, and numbers $r_i > 0$ such that $r_i \downarrow 0$ as $i \rightarrow \infty$, and

$$\mu(B(x, r_i) \cap H(x, \theta_i, \eta/2)) < \alpha_i h(r_i) \quad (2.22)$$

for all i . Suppose that (2.22) holds for x . By choosing a suitable subsequence, we may assume that $\theta_i \rightarrow \theta \in V \cap S^{n-1}$ as $i \rightarrow \infty$ and that $H(x, \theta, \eta) \subset H(x, \theta_i, \eta/2)$ for all i . Combined with (2.22), this completes the proof. \square

The last theorem of this section deals with purely m -unrectifiable sets. As mentioned before, it has been proved by Gillis [7] when $n = 2$ and by Lorent [9] when $m = n - 1$. We recall that a set $A \subset \mathbb{R}^n$ is called purely m -unrectifiable if $\mathcal{H}^m(A \cap E) = 0$ whenever E is a Lipschitz image of \mathbb{R}^m .

Theorem 2.11. *Suppose that $A \subset \mathbb{R}^n$ is purely m -unrectifiable with $\mathcal{H}^m(A) < \infty$. If $\eta > 0$, and $V \in G(n, m)$, then for \mathcal{H}^m -almost all $x \in A$, there is $\theta \in V \cap S^{n-1}$ such that $\underline{D}_m(\mathcal{H}^m \llcorner A, H(x, \theta, \eta), x) = 0$.*

If $m = 1$ and $\theta \in S^{n-1}$, then $\underline{D}_1(\mathcal{H}^1 \llcorner A, H(x, \theta, \eta), x) = 0$ for \mathcal{H}^1 -almost all $x \in \mathbb{R}^n$.

Proof. The proof is based on our proof of Theorem 2.1 and the Besicovitch-Federer projection theorem, see for example [16, theorem 18.1], which says that

$$\mathcal{H}^m(P_V(A)) = 0 \quad (2.23)$$

for $\gamma_{n,m}$ -almost all $V \in G(n, m)$.

By the Borel regularity of \mathcal{H}^m , we may assume that A is a Borel set. Then the assumptions of Theorem 2.1 are valid for $\mathcal{H}^m \llcorner A$ and $h(r) = r^m$, except for (h2). But assumption (h2) in Theorem 2.1 was only used to obtain (2.21) and if $V \in G(n, m)$ is such that (2.23) holds, then also (2.21) holds and the proof of Theorem 2.1 gives the assertion for V .

Now fix arbitrary $V \in G(n, m)$. According to the above facts, we may choose m -planes $V_i \in G(n, m)$ which fulfil (2.23) such that $d(V_i, V) \rightarrow 0$ in $G(n, m)$ as $i \rightarrow \infty$. For μ -almost every $x \in \mathbb{R}^n$, we may choose directions $\theta_i \in V_i \cap S^{n-1}$, and radii $r_i > 0$ such that

$$\mathcal{H}^m(B(x, r_i) \cap H(x, \theta_i, \eta/2) \cap A) / r_i^m \longrightarrow 0,$$

and $r_i \downarrow 0$ as $i \rightarrow \infty$. Using (1.1), we find a subsequence of $(\theta_i)_i$, which we denote by the same symbols, such that $\theta_i \rightarrow \theta \in V \cap S^{n-1}$ and $H(x, \theta, \eta) \subset H(x, \theta_i, \eta/2)$ for all i . The first assertion of the theorem follows.

The second statement can be verified by modifying the argument slightly. We omit the details, see also the discussion below. \square

As noted in the introduction, Besicovitch [2, theorem 13] proved that when $m = 1$ and $n = 2$, then one can take $\eta = 0$ in Theorem 2.11. His method can be modified to prove that this is true for all $n \in \mathbb{N}$. It remains unsolved, if this is true for $m > 1$.

Note that a simple compactness argument on S^{n-1} implies that the directions $\theta = \theta(x)$ can be chosen to be independent of η in Theorems 2.1 and 2.11.

We shall finish this section by the example that was mentioned earlier in this section. We perform a modification of a construction that has turned out to be useful in many connections related to fractal sets and measures, see for example [15, example 4.4] or [8, example 3.1].

Example 2.12. There exists a nondecreasing function $h:]0, 1[\rightarrow]0, \infty[$ which fulfils conditions (h1) and (h2) of Theorem 2.1 and a Radon measure μ on \mathbb{R}^n such that for all $\theta \in S^{n-1}$, $0 < \eta < 1$, and for μ -almost every $x \in \mathbb{R}^n$,

$$\underline{D}_h(\mu, x) < \infty, \tag{2.24}$$

$$\underline{D}_h(\mu, H(x, \theta, \eta), x) > c, \tag{2.25}$$

where $c > 0$ is a constant depending on η .

Construction. We assume that $n = 1$. A similar construction works also in higher dimensions. The idea is to perform a Cantor-type construction resulting to a measure μ with $0 < \underline{D}_{1/2}(\mu, x) < \infty$ and $\overline{D}_{1/2}(\mu, x) = \infty$ for μ -almost every x on \mathbb{R} . The conditions (2.24) and (2.25) are obtained by defining h so that at some scales $h(r)$ behaves like $r^{1/2}$, whereas at some other scales $h(r)$ is very much smaller than $r^{1/2}$.

Define numbers q_k and l_k for $k = 2, 3, 4, \dots$ by setting $q_k = 1 - 1/k^2$ and $l_k = k^6$. It is a simple matter to check that

$$\prod_{k=2}^{\infty} q_k > 0, \tag{2.26}$$

$$1 - q_k = k/l_k^{1/2}. \tag{2.27}$$

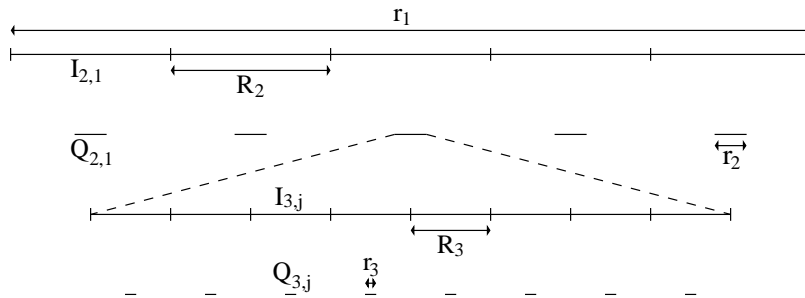


FIGURE 5. Construction of the intervals $I_{i,j}$ and $Q_{i,j}$. The biggest interval on the top is $Q_{1,1}$.

We also define numbers r_k and R_k for $k \in \mathbb{N}$ by setting $R_1 = 1$, $r_k = R_k/l_{k+1}$, and $R_{k+1} = r_k/l_{k+1} = R_k/l_{k+1}^2$. The above definitions imply that

$$l_2 \cdots l_k R_k^{1/2} = 1. \quad (2.28)$$

See figure 5 to get idea of the following construction. Let $I_{1,1} = [0, 1]$ and $Q_{1,1} = [1/2 - r_1/2, 1/2 + r_1/2]$. Divide $Q_{1,1}$ into l_2 closed subintervals of equal length and denote them by $I_{2,1}, \dots, I_{2,l_2}$. Note that the length of these intervals is R_2 . For every $k = 1, \dots, l_2$, let $Q_{2,k}$ be the closed interval with same centre as $I_{2,k}$ and with length $r_2 = R_2/l_3$. Suppose that closed intervals $Q_{k,i}$ have been defined for every $i = 1, \dots, l_2 \cdots l_k$. Divide every interval $Q_{k,i}$ into l_{k+1} closed subintervals of equal length and denote all these intervals by $I_{k+1,1}, \dots, I_{k+1,l_2 \cdots l_{k+1}}$. For every i , let $Q_{k+1,i} \subset I_{k+1,i}$ be a closed interval with same centre as $I_{k+1,i}$ and with length r_{k+1} . For every integer $k \geq 2$, this construction gives $l_2 \cdots l_k$ closed intervals $I_{k,i}$ and $Q_{k,i}$ with lengths R_k and r_k , respectively.

We now define h and μ . For $k \in \mathbb{N}$ let

$$h(r) = \begin{cases} r_k^{1/2} & \text{if } r_k \leq r < R_k, \\ r^{1/2} & \text{if } R_{k+1} \leq r \leq r_k. \end{cases} \quad (2.29)$$

The measure μ is defined on the set

$$A = \bigcap_{k=2}^{\infty} \bigcup_{i=1}^{l_2 \cdots l_k} I_{k,i}$$

by repeated subdivision, that is

$$\mu(A \cap I_{k,i}) = \mu(A \cap Q_{k,i}) = 1/(l_2 \cdots l_k). \quad (2.30)$$

This measure clearly extends to a Radon probability measure on \mathbb{R} , see for example [5, proposition 1.7].

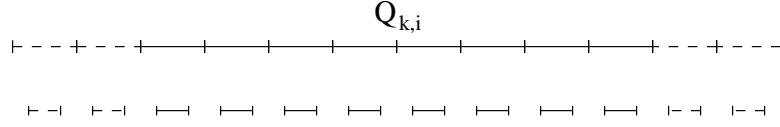


FIGURE 6. Construction of the set B_p . The dashed intervals are left outside B_p .

Clearly $\lim_{r \downarrow 0} h(r) = 0$ and further (recall (2.28))

$$\begin{aligned} \liminf_{r \downarrow 0} h(r)/r &= \lim_{k \rightarrow \infty} r_k^{1/2}/R_k = \lim_{k \rightarrow \infty} 1/(l_{k+1}R_k)^{1/2} \\ &= \lim_{k \rightarrow \infty} l_2 \cdots l_k / l_{k+1}^{1/2} = \lim_{k \rightarrow \infty} (k!)^6 / (k+1)^3 = \infty. \end{aligned}$$

Thus h satisfies (h1) and (h2). By (2.28) and the definition of μ , we deduce that if $x \in A$, then

$$\liminf_{r \downarrow 0} \frac{\mu(B(x, r))}{h(r)} \leq \limsup_{k \rightarrow \infty} \frac{\mu(B(x, R_k))}{h(R_k)} \leq \frac{3}{l_2 \cdots l_k R_k^{1/2}} = 3 < \infty$$

and so also (2.24) is valid.

It remains to verify (2.25). For every $p \in \mathbb{N}$ we define a set $B_p \subset A$ so that the first p stages of the construction of B_p are similar as those of A , but after this some (small) amount of intervals $I_{k,i}$ are left outside B_p at each stage. To be precise, when we divide $Q_{k,i}$ ($k \geq p$) into l_{k+1} subintervals, we leave $\lfloor l_{k+1}(1 - q_{k+1})/2 \rfloor$ (we use notation $\lfloor a \rfloor = \max\{n \in \mathbb{N} : n \leq a\}$) leftmost and also $\lfloor l_{k+1}(1 - q_{k+1})/2 \rfloor$ rightmost of them outside B_p , see figure 6. Formally (with a suitable enumeration of the intervals $I_{k,i}$),

$$B_p = \bigcap_{k=p+1}^{\infty} \bigcup_{i=1}^{i_k} I_{k,i},$$

where

$$i_k = l_2 \cdots l_p \prod_{j=p+1}^k (l_j - 2\lfloor l_j(1 - q_j)/2 \rfloor) \geq l_2 \cdots l_k \prod_{j=p+1}^k q_j.$$

It follows from the above formula and (2.26), that $\mu(B_p) \rightarrow 1 = \mu(\mathbb{R})$ as $p \rightarrow \infty$.

Fix $x \in B_p$ and assume that $\theta = 1$. The case $\theta = -1$ can be handled similarly. Let $R_{k+1} \leq r < R_k$ for some $k \geq p$. Now x belongs to one of the intervals $Q_{k,i}$ which we denote by $[\alpha, \beta]$. Let $d = \beta - x$. Recall that then $d \leq r_k$.

Suppose first that $d \leq r < R_k$, then by the definition of B_p , the interval $[x, x+r]$ contains at least $\lfloor l_{k+1}(1 - q_{k+1})/2 \rfloor$ intervals from the collection $\{I_{k+1,i}\}_i$, see figure 7. We compute, using (2.30), (2.29), (2.27), and (2.28):

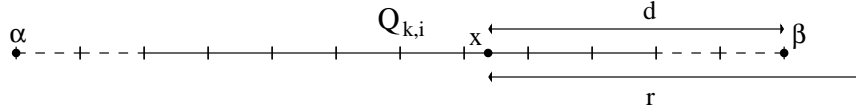


FIGURE 7

$$\begin{aligned}
\mu([x, x+r])/h(r) &\geq [2^{-1}l_{k+1}(1-q_{k+1})] (l_2 \cdots l_{k+1})^{-1} r_k^{-1/2} \\
&\geq \left(2^{-1}(k+1)l_{k+1}^{1/2} - 1\right) (l_2 \cdots l_k)^{-1} l_{k+1}^{-1} r_k^{-1/2} \\
&\geq 2^{-1}k (l_2 \cdots l_k)^{-1} l_{k+1}^{-1/2} r_k^{-1/2} \\
&= 2^{-1}k (l_2 \cdots l_k)^{-1} R_k^{-1/2} = k/2.
\end{aligned}$$

Suppose then that $R_{k+1} \leq r < d$. If $r \in [mR_{k+1}, (m+1)R_{k+1}]$ for some $m \in \mathbb{N}$, then (recall (2.30)) $\mu([x, x+r]) \geq m/(l_2 \cdots l_{k+1})$. Therefore $\mu([x, x+r]) \geq r/(2R_{k+1}l_2 \cdots l_{k+1})$. This together with (2.29) and (2.28) gives

$$\begin{aligned}
\mu([x, x+r])/h(r) &= \mu([x, x+r])/r^{1/2} \\
&\geq (r/R_{k+1})^{1/2} / \left(2l_2 \cdots l_{k+1}R_{k+1}^{1/2}\right) \geq 1/2.
\end{aligned}$$

We conclude that $\underline{D}_h(\mu, H(x, 1), x) \geq 1/2$ for all $x \in B_p$. \square

3. UPPER DENSITIES

Theorem 1.2, and also the upper density results of Salli [18, theorems 3.1, 3.7 and 3.8] and Mattila [14, theorem 3.3], are readily generalised for measures μ that satisfy $0 < \overline{D}_s(\mu, x) < \infty$ for μ -almost every $x \in \mathbb{R}^n$. This arises from the fact that this kind of measures behave in small scales very much like \mathcal{H}^s , see [16, theorem 6.9]. Of course, one has to replace the constants $c(n, m, \eta)$ and $c(n, s, \eta)$ in Theorem 1.2 by $\overline{D}_s(\mu, x)c(n, m, \eta)$ and $\overline{D}_s(\mu, x)c(n, s, \eta)$, respectively.

Our goal in this section is to give a generalisation of Theorem 1.2 that applies for measures which satisfy μ -almost everywhere the condition $\underline{D}_h(\mu, x) < \infty$, provided that the function h fulfils some additional conditions. Our technique is adapted from Salli [18, theorem 3.1]. Notice that we may well have $\overline{D}_h(\mu, x) = \infty$ for μ -almost every $x \in \mathbb{R}^n$.

Theorem 3.1. *Let $h:]0, r_0[\rightarrow]0, \infty[$ be a function which satisfies, for some $0 < c_1 < \infty$, the doubling condition*

$$h(2r) < c_1 h(r). \quad (\text{d})$$

Assume also that for some $m \in \mathbb{N}$ and for all $r < r_0$, $0 < t < 1$:

$$h(tr) \leq t^m h(r), \quad (\text{h4})$$

$$\lim_{r \downarrow 0} h(r)/r^m = 0. \quad (\text{h5})$$

If μ is a Borel measure on \mathbb{R}^n such that $\underline{D}_h(\mu, x) < \infty$ for μ -almost all $x \in \mathbb{R}^n$, $V \in G(n, n - m)$, and $0 < \eta \leq 1$, then

$$\overline{D}_h(\mu, X(x, V, \eta), x) \geq c \overline{D}_h(\mu, x)$$

for μ -almost all $x \in \mathbb{R}^n$. Above the constant $c > 0$ depends only on m, n, η , and c_1 .

If $m < s < n$, then $h(r) = r^s$ clearly satisfies conditions (d), (h4), and (h5). So the statement of Theorem 3.1 holds for $\mu = \mathcal{P}^s \llcorner A$, when $m < s < n$, $h(r) = r^s$, and $A \subset \mathbb{R}^n$ has finite \mathcal{P}^s measure. Theorem 3.1 can also be applied to measures \mathcal{H}_h and \mathcal{P}_h provided that h fulfils the required assumptions, recall (1.2) and (1.3). One can take, for example, $h(r) = r^s \log(1/r)$, where $m < s < n$, or $h(r) = r^m / \log(1/r)$.

When talking about cubes in \mathbb{R}^n we shall, hereafter, mean sets of the form $\{x\} + [-r, r]^n$, where $x \in \mathbb{R}^n$, and $r > 0$. In the following proofs $d(A)$ stands for the diameter of the set $A \subset \mathbb{R}^n$.

Proof of Theorem 3.1. It follows from doubling condition (d) that there is a constant $c_2 < \infty$ such that

$$h\left(2\left(n - m + \eta^2/16\right)^{1/2} r\right) < c_2 h(r) \quad (3.1)$$

when $r > 0$ is small enough. We can assume that this is true for all $0 < r < r_0$. Define c by

$$c = 2^{-1} c_2^{-1} (4m^{1/2} \eta^{-1} + 1)^{-m}. \quad (3.2)$$

Then $c \geq c'(n, m, c_1) \eta^m$.

We may assume that $V = \{x : \text{proj}_i x = 0 \text{ for all } i = n - m + 1, \dots, n\}$. Fix $M > 0$ and define

$$B = \{x \in \mathbb{R}^n : \overline{D}_h(\mu, x) > M \text{ and } \underline{D}_h(\mu, x) < \infty\}.$$

It is sufficient to show that $\overline{D}_h(\mu, X(x, V, \eta), x) \geq cM$ for μ -almost all $x \in B$. We will show that for any $\alpha > 0$, the set

$$F = \{x \in B : \mu(B(x, r) \cap X(x, V, \eta)) \leq cMh(r) \text{ for all } 0 < r < \alpha\}$$

is of μ measure zero. Assume on the contrary that $\mu(F) > 0$. One can use quite standard methods, see Lemma 2.7, to show that F is a Borel set, and hence we can assume it to be closed.

Fix $k \in \mathbb{N}$ such that

$$4m^{1/2}/k < \eta \leq 4m^{1/2}/(k-1). \quad (3.3)$$

According to Lemma 2.6 (2), there is $x_0 \in F$ such that $\overline{D}_h(\mu, F, x_0) > M$. We may assume that $x_0 = 0$. We can now fix $r_1 > 0$ such that

$$\mu([-r_1, r_1]^n \cap F) > Mh(r_1).$$

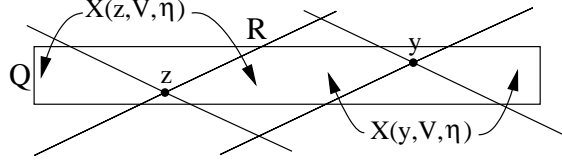


FIGURE 8

We will next select recursively cubes $R_j \subset \mathbb{R}^{n-m}$ and $Q_j \subset \mathbb{R}^m$. Let $R_0 = [-r_1, r_1]^{n-m}$ and $Q_0 \subset [-r_1, r_1]^m$ such that $\ell(Q_0) = 2r_1/k$ ($\ell(Q)$ denotes the side length of Q) and $\mu((R_0 \times Q_0) \cap F) > Mh(r_1)/k^m$. Such a cube exists, because $[-r_1, r_1]^m$ can be divided into k^m cubes having side-length $2r_1/k$. Suppose that cubes R_j and Q_j have been selected. Denote by S one of the minimal cubes $S \subset R_j$ such that $P_V((R_j \times Q_j) \cap F) \subset S$. If

$$\ell(S) > \ell(R_j)/2, \quad (3.4)$$

then we finish our selection. Otherwise we select $R_{j+1} \subset R_j$ such that $S \subset R_{j+1}$ and $\ell(R_{j+1}) = \ell(R_j)/2$. We also choose $Q_{j+1} \subset Q_j$ such that $\ell(Q_{j+1}) = \ell(Q_j)/2$ and $\mu((R_{j+1} \times Q_{j+1}) \cap F) \geq 2^{-m}\mu((R_j \times Q_j) \cap F)$.

If j is an index such that R_j and Q_j are selected, then

$$\mu((R_j \times Q_j) \cap F) > 2^{-jm} M k^{-m} h(r_1) \quad (3.5)$$

and (recall (3.3))

$$\begin{aligned} d(R_j \times Q_j) &= 2^{-j} d(R_0 \times Q_0) = 2^{-j+1} (n + m(k^{-2} - 1))^{1/2} r_1 \\ &< 2^{-j+1} (n - m + \eta^2/16)^{1/2} r_1. \end{aligned} \quad (3.6)$$

The next step is to show that for some index j_0 our selection comes to an end. If this is not the case, let $\{x_1\} = \bigcap_{j=1}^{\infty} (R_j \times Q_j) \cap F$, and denote $d_j = d(R_j \times Q_j)$. Let $d_{j+1} \leq r < d_j$. Using (3.5), and (3.6), we get

$$\begin{aligned} \mu(B(x_1, r))/h(r) &\geq \mu((R_{j+1} \times Q_{j+1}) \cap F)/h(d_j) \\ &> 2^{-(j+1)m} M k^{-m} h(r_1)/h\left(2^{-j+1} (n - m + \eta^2/16)^{1/2} r_1\right) \end{aligned}$$

It follows from (h5), that the right hand side of the above inequality tends to infinity as $j \rightarrow \infty$. This yields $\underline{D}_h(\mu, x_1) = \infty$, which is impossible since $x_1 \in B$.

Thus, by the above argument, there is an index j_0 such that (3.4) holds. We abbreviate $R = R_{j_0}$, $Q = Q_{j_0}$ and $d = d_{j_0}$. Pick $y, z \in (R \times Q) \cap F$ such that $\text{proj}_i(y - z) > \ell(R)/2$ for some $i \in \{1, \dots, n - m\}$. We will show that (see figure 8)

$$R \times Q \subset X(y, V, \eta) \cup X(z, V, \eta). \quad (3.7)$$

Let $x \in R \times Q$. If $\text{proj}_i(x) \geq \text{proj}_i(y + z)/2$, then

$$|x - z| \geq \text{proj}_i(x - z) \geq \text{proj}_i(y - z)/2 > \ell(R)/4 = 2^{-j_0-1} r_1.$$

Using this and (3.3), we deduce

$$d(x - z, V) = P_{V^\perp}(x - z) \leq d(Q) = 2^{-j_0+1} r_1 k^{-1} m^{1/2} < \eta |x - z|.$$

Hence $x \in X(z, V, \eta)$. If $\text{proj}_i(x) < \text{proj}_i(y + z)/2$, then by symmetry $x \in X(y, V, \eta)$. Therefore (3.7) holds and we know that with $x = y$ or $x = z$ we have

$$\mu(X(x, V, \eta)) \geq \mu(R \times Q)/2. \quad (3.8)$$

Finally we compute, using (3.1)–(3.8), and (h4),

$$\begin{aligned} & \mu(X(x, V, \eta) \cap B(x, d)) / h(d) \\ & \geq 2^{-1} \mu((R \times Q) \cap F) / h(d) \\ & > M k^{-m} 2^{-j_0 m - 1} h(r_1) / h\left((n - m + \eta^2/16)^{1/2} 2^{-j_0+1} r_1\right) \\ & \geq M k^{-m} 2^{-1} c_2^{-1} \\ & \geq c M. \end{aligned}$$

This contradicts the definition of F . \square

When $m = n - 1$ it is natural to ask whether one can replace the cones $X(x, V, \eta)$ in the above theorem by their one half. The next theorem shows that the answer is positive, at least with a little bit stronger assumptions on h . For example the functions $h(r) = r^s \log(1/r)$, where $n - 1 < s < n$, satisfy assumption (h6) of the following theorem.

Theorem 3.2. *Assume that $h:]0, r_0[\rightarrow]0, \infty[$ is a function satisfying doubling condition (d). Suppose also that for some $s > n - 1$ and for all $r < r_0$, $0 < t < 1$,*

$$h(tr) \leq t^s h(r). \quad (\text{h6})$$

If μ is a Borel measure on \mathbb{R}^n such that $\underline{D}_h(\mu, x) < \infty$ for μ -almost all $x \in \mathbb{R}^n$, $0 < \eta \leq 1$, and $\theta \in S^{n-1}$, then

$$\overline{D}_h(\mu, X^+(x, \theta, \eta), x) \geq c \overline{D}_h(\mu, x)$$

for μ -almost all $x \in \mathbb{R}^n$, where the constant $c > 0$ depends only on n, s, η , and c_1 (see (d)).

Proof. Define c_2 as in the previous proof and set

$$c = c_2^{-1} (1 - 2^{-(s-n+1)/2}) (4(n-1)\eta^{-1} + 1)^{1-n}.$$

Then $c \geq c'(n, c_1)(s - n + 1)\eta^{n-1}$. We may assume that $\theta = e_1$. Fix $M > 0$ and define sets B and F , numbers k and r_1 , and point $x_0 \in \mathbb{R}^n$ in a corresponding manner as in the previous proof. We assume again, to simplify notation, that $x_0 = 0$. We begin to choose intervals $I_j \subset \mathbb{R}$ and cubes $Q_j \subset \mathbb{R}^{n-1}$ by setting $I_0 = [-r_1, r_1]$ and selecting $Q_0 \subset [-r_1, r_1]^{n-1}$ such that $\ell(Q_0) = 2r_1/k$ and $\mu((I_0 \times Q_0) \cap F) > M k^{1-n} h(r_1)$. Assume that I_j and Q_j have been selected.

Let $a_j = \inf \text{proj}_1 ((I_j \times Q_j) \cap F)$ and $b_j = \sup \text{proj}_1 ((I_j \times Q_j) \cap F)$. If the conditions

$$b_j - a_j > \ell(I_j) / 2, \quad (3.9)$$

$$\mu(((a_j + b_j) / 2, b_j] \times Q_j) \cap F) > (1 - 2^{-(s-n+1)/2}) \mu((I_j \times Q_j) \cap F) \quad (3.10)$$

hold, then we finish our selection. If (3.9) is not valid, then we choose $I_{j+1} \subset I_j$ such that $[a_j, b_j] \subset I_{j+1}$ and $\ell(I_{j+1}) = \ell(I_j) / 2$. We also take $Q_{j+1} \subset Q_j$ such that $\ell(Q_{j+1}) = \ell(Q_j) / 2$ and

$$\mu((I_{j+1} \times Q_{j+1}) \cap F) \geq 2^{1-n} \mu((I_j \times Q_j) \cap F).$$

If (3.9) holds but (3.10) does not, then we select $I_{j+1} \subset I_j$ such that $\ell(I_{j+1}) = \ell(I_j) / 2$, and

$$\mu((I_{j+1} \times Q_j) \cap F) \geq 2^{-(s-n+1)/2} \mu((I_j \times Q_j) \cap F),$$

and further $Q_{j+1} \subset Q_j$ such that $\ell(Q_{j+1}) = \ell(Q_j) / 2$ and

$$\mu((I_{j+1} \times Q_{j+1}) \cap F) \geq 2^{1-n} 2^{-(s-n+1)/2} \mu((I_j \times Q_j) \cap F).$$

For every I_j and Q_j , one obtains

$$\begin{aligned} \mu((I_j \times Q_j) \cap F) &> 2^{j(-n-s+1)/2} M k^{1-n} h(r_1), \\ d(I_j \times Q_j) &= 2^{-j} d(I_0 \times Q_0) < 2^{-j+1} (1 + \eta^2/16)^{1/2} r_1. \end{aligned}$$

Similar reasoning as in the proof of Theorem 3.1 combined with assumption (h6) yields that our process of selecting intervals I_j and cubes Q_j must terminate. Thus, for some index j_0 , sets I_{j_0} and Q_{j_0} satisfy (3.9) and (3.10). Abbreviate $a = a_{j_0}$, $b = b_{j_0}$, and so on. Let $y \in (I \times Q) \cap F$ be such that $\text{proj}_1 y = a$. Then

$$[(a + b) / 2, b] \times Q \subset X^+(y, \theta, \eta) \cap B(y, d).$$

We obtain, as in the proof of Theorem 3.1, that

$$\mu(X^+(y, \theta, \eta) \cap B(y, d)) / h(d) \geq c M.$$

This leads to a contradiction. □

Let $0 < \alpha < \infty$. It is clear from the proofs that assumptions (h4) and (h6) in Theorems 3.1 and 3.2 can be weakened to $h(tr) \leq \alpha t^m h(r)$ and $h(tr) \leq \alpha t^s h(r)$, respectively. It also suffices to assume that these conditions hold when $0 < t < \beta$ for some $\beta > 0$. However, under these slightly weaker assumptions, constant c will depend also on numbers α and β .

On the real line the question of upper densities is easier. We state the following theorem without a proof. A somewhat similar calculation as that of [13, theorem 7] applies.

Theorem 3.3. *If μ is a locally finite Borel measure on \mathbb{R} , then for any $h:]0, \infty[\rightarrow]0, \infty[$,*

$$\overline{D}_h(\mu, [x, \infty[, x) = \overline{D}_h(\mu,] - \infty, x], x) \geq \overline{D}_h(\mu, x) / 2$$

for μ -almost every $x \in \mathbb{R}$.

The local finiteness of μ in the above result can be replaced, for example, by assuming that $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$, and $\underline{D}_h(\mu, x) < \infty$ for μ -almost all $x \in \mathbb{R}$.

If μ is a measure on \mathbb{R}^n , $A \subset \mathbb{R}^n$, and $x \in \mathbb{R}^n$, we define

$$\overline{D}_\mu(A, x) = \limsup_{r \downarrow 0} \mu(B(x, r) \cap A) / \mu(B(x, r)).$$

If $0 < \overline{D}_h(\mu, x) < \infty$ for μ -almost every $x \in \mathbb{R}^n$ in Theorems 3.1 and 3.2, then they can be used to obtain positive lower bounds for the upper densities $\overline{D}_\mu(X(x, V, \eta), x)$ and $\overline{D}_\mu(X^+(x, \theta, \eta), x)$. Under the same condition, theorem 3.3 gives positive lower bounds for $\overline{D}_\mu([x, \infty[, x)$ and $\overline{D}_\mu(] - \infty, x], x)$.

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